

Dynamical Adjustment of Propagators in Renormalization Group Flows

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July, 2006

Abstract

A class of continuous renormalization group flows with a dynamical adjustment of the propagator is introduced and studied theoretically for fermionic and bosonic quantum field theories. The adjustment allows to include self-energy effects nontrivially in the denominator of the propagator and to adapt the scale decomposition to a moving singularity, and hence to define flows of Fermi surfaces in a natural way. These flows require no counterterms, but the counterterms used in earlier treatments can be constructed using them. The influence of propagator adjustment on the strong-coupling behaviour of flows is examined for a simple example, and some conclusions about the strong coupling behaviour of renormalization group flows are drawn.

1 Introduction

The renormalization group (RG) method provides a way to attack, and in many cases solve, the infrared problem of models in field theory, statistical mechanics

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[†]Supported in part by DFG grant Sa 1362/1-1, NSERC and an ESI senior fellowship

and many-body theory. Infrared problems are ubiquitous and physically natural, since they are inherent to systems with gapless excitations, such as critical systems of statistical mechanics, massless field theories, and systems with spontaneously broken continuous symmetries.

In systems of correlated fermions modelling metals, excitations around the Fermi surface are gapless. The attempt to treat interaction effects by naive perturbation theory leads to a severe infrared problem in the form of divergences due to small denominators. The RG has been used for weak, short-range two-body interactions to cure this problem in perturbation theory and to classify the interaction terms by their power counting degree [1, 2]. The one-body (or self-energy) terms, quadratic in the fermion fields, are the relevant ones, followed by the marginally relevant parts of the two-body interaction, which are quartic in the fermion fields. All higher terms are power counting irrelevant at weak coupling. The self-energy terms change the Fermi surface, which is the location of the singularity of the fermion propagator. Because the singularity has to be kept track of precisely, neglecting these terms leads to problems even in formal perturbation theory (provided it is done to higher than second order). For this reason, the self-energy terms are the most relevant ones. The marginally relevant parts of the two-body terms determine the quasiparticle interactions and give information about ordering tendencies and collective phenomena. The hierarchy of equations couples the evolution of the different terms.

By the above arguments, a possible way to proceed is to assume that the Fermi surface given by the free propagator is already that of the interacting system. One can then take the dispersion function of a certain form, hence the interacting Fermi surface of a certain shape, assuming that this can be realized in an interacting system, and treat the remaining selfenergy effects by keeping them in the form of two-legged vertices. This also involves the assumption that there is no large correction to the quasiparticle weight. Both assumptions can, with considerable technical effort, be verified for the situation of smooth, positively curved Fermi surfaces [2, 3, 10]. Technically, this is done in [2] using counterterms to fix the Fermi surface. The counterterm function determines the shift from the Fermi surface of the free to that of the interacting system. The above realizability question takes the form of an inversion problem, which is solved under certain conditions in [2].

Although the importance of the self-energy terms is known and clearly acknowledged in the literature, a way to proceed in applied studies has been to focus on the study of the two-body interactions while neglecting self-energy terms in the flow. The main motivation for this is simplicity. Controlling the relevant self-

energy terms carefully is technically complicated already in the case of a smooth Fermi surface. The counterterm method of [1, 2] provides a conceptually clear way of dealing with the problem, but it has been used only in a few applied studies so far.

In the two-dimensional Hubbard model, the case with Van Hove singularities is of interest for high- T_c superconductivity [4]. For this case, a renormalized expansion using counterterms is developed, and some regularity properties of the self-energy are shown, in [5]. However, the question of realizability of singular Fermi surface shapes has not been fully answered yet, although it is very important for deciding how generic Van Hove singularities really are in two-dimensional interacting systems [5]. The effects of Fermi surface dynamics near Van Hove points can be rather nontrivial, since the Fermi surface is most easily moved in the vicinity of a zero of the gradient of the dispersion relation. This point was first clearly realized in [6], and it was shown that quantum fluctuations of the Fermi surface can arise and lead to non-Fermi liquid behaviour [7]. In this situation, it is useful to have a method in which the adjustment of the Fermi surface is done in the flow itself, because the concept of a non-fluctuating Fermi surface itself may make sense only down to a certain scale, below which the fluctuations emerge and may become dominant.

The method discussed in this paper allows to use dynamically adjusting scale decompositions and to study Fermi surface flows. As a method, it applies more generally and can be used flexibly to adapt and adjust propagators in bosonic and fermionic (or mixed) systems, both for continuous and discrete flows. Variants of the method are used in mathematical studies of Fermi surface flows and Fermi liquid theory [3]. Also, the method has already been used to calculate the Fermi surface flow for the Hubbard model (see the Appendix of [8]).

Various renormalization group schemes have already been used to study correlated fermion systems: Polchinski's original scheme [9], the Wick ordered scheme [10, 11], the 1PI scheme [12, 13, 8]. Flow equations for the 2PI functions were introduced [14] and studied for quasi-1d systems [15]. In this context one may well ask why one would want to investigate another scheme. In the following, I give some further motivations for this.

As is well-known, all the above-mentioned schemes are equivalent on the level of untruncated hierarchies, because all of them contain the full information about all Green functions, as long as they are kept as infinite hierarchies. But they start to differ once the approximations that are necessary in nontrivial applications, are made, most importantly truncations of infinite hierarchies, gradient expansions of the generating functionals, or simplifying assumptions about the dependence

of the Green functions or vertices on momenta and frequencies. It is then useful to understand which features of a given scheme make it useful for a particular situation.

All of the above-mentioned schemes include, in their untruncated form, all self-energy effects in the flow, albeit in a rather different way. In their original form, the Polchinski and Wick ordered schemes keep self-energy effects as vertices, while the 1PI scheme includes them in the most natural way, namely by having full propagators on the lines. On the other hand, the Wick ordered scheme can be transformed so as to have full propagators on the lines (see Section 4.7.2 of [16]), and a similar transformation also applies to the Polchinski scheme. However, the standard momentum space setup of the RG with a cutoff in momentum space near to the Fermi surface does not include an adaptive scale decomposition, and thus does not allow for the above-mentioned flow of the Fermi surface. This is the case in all schemes, so a correct procedure would require using counterterms even in the 1PI scheme (in most applied studies, the Fermi surface shift was neglected, so that this problem did not appear). Alternatively, one can try not to use a cutoff on the fermion momentum [17, 18], but power counting has been done rigorously in more than one dimension only in the momentum space scheme [1, 2, 10].

In two- or higher-dimensional applications, it is a serious practical problem to take the momentum and frequency dependence of the flowing vertex functions into account. The full propagator G of the 1PI scheme has support on all scales above the cutoff scale. This makes it necessary to extend the loop integrals over all of momentum space, which in turn requires having an accurate form of G , and of the higher vertex functions, over a wide range of momenta and frequencies. This has not been achieved so far. The situation in the other schemes is better in this respect: in the Polchinski equation, all internal lines carry a derivative of the cutoff function. Therefore, at scale ϵ , the propagators get evaluated only at values of the momentum k with $|e(k)| \approx \epsilon$. The integration procedure for Polchinski's equation makes the equation nonlocal in the scale parameter in the standard way [19], but a significant advantage is that whenever a momentum region $|e(k)| \approx \epsilon'$ further away from the Fermi surface contributes to a loop integral in the flow equation, it does so only via a propagator on scale near to ϵ' , which was already calculated in an earlier stage of the flow, and which does not need to be adjusted in every further integration step. In the Wick ordered scheme, all propagators are supported at or below the cutoff, so only small neighbourhoods of the Fermi surface occur in the loop integrals at low scales. Thus the propagators and vertices are needed only close to the Fermi surface and at small frequencies.

In studies and talks by the Göttingen group [20], it was pointed out that in simple models, the 1PI scheme has a better strong coupling behaviour than the other schemes. Namely, the 1PI flow equation is accurate in these toy models at much larger coupling values than one might believe at first, far beyond the weak coupling regime, and it remains accurate in a region of intermediate to strong couplings where all other schemes fail. The dynamically adjusting scheme is examined for this toy model in Section 4. It turns out that propagator adjustment, namely putting selfenergy effects into the denominator instead of keeping them as vertices, is a necessary, but not sufficient, ingredient in stabilizing the equations at strong coupling. The Polchinski scheme still fails at strong coupling, while the dynamically adjusted Wick ordered scheme is accurate up to arbitrarily large couplings. The analysis also clarifies the reasons for the success of the 1PI scheme. In Section 4, I also give a more general discussion about strong initial couplings, and show that for reasons that are simple, but specific to the RG method, the “weak-coupling” RG flow is a well-defined starting point at strong coupling also in “real” models (such as the two-dimensional Hubbard model), contrary to what one may believe at first, and to what is stated in many places. Whether this can be used efficiently in computations depends on certain relative rates of growth; this is under investigation and will be discussed at the end.

2 Theory

2.1 Setup

The partition function of a general bosonic or fermionic theory with source fields Ψ (the generating function for the Green functions) is

$$P_0(\Psi) = \mathcal{N} \int \mathcal{D}\Theta \, e^{-\frac{1}{2}(\Theta, A_0 \Theta) - \mathcal{V}_0(\Theta) + (\Psi, \Theta)}. \quad (1)$$

Here the fields Θ_X are indexed by elements X of a finite set \mathcal{X} . \mathcal{X} is taken finite to make the generating function mathematically well-defined. In many applications, $\mathcal{X} = \Gamma \times I$, where Γ is a finite space-time lattice and I is a finite set of internal indices, but this assumption will not be needed for the general theory. If \mathcal{X} comes from such a lattice regularization of a continuum theory, the limit of lattice spacing to zero and volume to infinity can be taken later in the RG equations (and after a volume factor has been divided out in the field-independent term). Readers who trust that all this can be done properly (or who don’t care) can also think of a continuous space Γ right away.

For bosons, the Θ_X are real-valued; for fermions, each Θ_X is one of the generators of a Grassmann algebra. Spinors and vector fields are included in this setting by an appropriate choice of I , and complex bosonic fields as well, by regarding them as two-component real fields (with the corresponding index again part of I). The commutation relations are

$$\Theta_X \Theta_Y = \zeta \Theta_Y \Theta_X \quad (2)$$

with $\zeta = 1$ for bosons and $\zeta = -1$ for fermions. The integration measure is $\mathcal{D}\Theta = \prod_{X \in \mathcal{X}} d\Theta_X$, and $(\Psi, \Theta) = \alpha \sum_{X \in \mathcal{X}} \Psi_X \Theta_X$ with $\alpha > 0$; (when considering a lattice approximation, α is the volume of the unit cell of the lattice). The action of the operator A_0 is $(A_0 \Theta)_X = \alpha \sum_Y (A_0)_{XY} \Theta_Y$; the matrix elements $(A_0)_{X,Y}$ are in general complex. An operator C is called ζ -symmetric if

$$(\Psi, C\Theta) = \zeta (C\Psi, \Theta), \quad (3)$$

(i.e. $C_{X,Y} = \zeta C_{Y,X}$). In the bosonic case, $\text{Re } C$ is also required to have nonnegative eigenvalues, so that for all Θ ,

$$(\Theta, (\text{Re } C)\Theta) \geq 0 \quad (4)$$

(in the case of complex fields viewed as two-component real fields, this corresponds to the usual condition that the hermitian part of the covariance is positive definite). Assume that the A_0 in (1) is ζ -symmetric. If A_0 is invertible, both A_0 and the *propagator* $C = A_0^{-1}$ are ζ -symmetric. The normalization factor \mathcal{N} is chosen such that $P_0(0) = 1$ for $\mathcal{V} = 0$. For bosons, I also assume that \mathcal{V} is bounded below and that $\frac{1}{2}(\Theta, A_0 \Theta) + \mathcal{V}(\Theta)$ grows faster than linearly in all $|\Theta_X|$ for large $|\Theta_X|$. Under these hypotheses, the integral in (1) is a convergent finite-dimensional integral in the case of bosons. In the case of fermions, it is a linear functional on a finite-dimensional space. The theory can also contain both bosonic and fermionic fields. Every formal functional integral can be regularized so that it takes this form.

The logarithm of P_0 generates the connected Green functions of the theory. Assume that A_0 is invertible and let $C = A_0^{-1}$. A shift in the integration variable gives (using the symmetry of the quadratic part under $\Theta \rightarrow -\Theta$)

$$P_0(\Psi) = e^{\frac{1}{2}(C\Psi, \Psi)} \int d\mu_C(\Theta) e^{-\mathcal{V}_0(\Theta + C\Psi)} \quad (5)$$

with $d\mu_C(\Theta)$ the normalized Gaussian measure with covariance C . In this form, one can take a limit where some of the eigenvalues of C go to zero; in the bosonic

case this corresponds to restricting the integral to the subspace $\ker C$. Typically, this situation arises by introducing cutoff functions that vanish strictly in parts of momentum space; see the following examples.

Although the equations derived in this paper work for the general case set up above, it is useful to keep two prototypical examples in mind.

The first example is scalar field theory on a finite lattice $\Gamma \subset \mathbb{Z}^3$ where $|I| = 1$, (A_0) is the discrete Laplacian with (say) Dirichlet boundary conditions, and $\mathcal{V}_0(\Theta) = \sum_{X \in \Gamma} V(\Theta_X)$, with V a polynomial that is bounded below. In that case, $(A_0)_{X,Y} = a_0(X - Y)$ with Fourier transform $\hat{a}_0(k) = 2 \sum_{i=1}^3 (1 - \cos k_i)$ for $k \neq 0$. The momentum $k = 0$ is removed by the Dirichlet condition; of course, other boundary conditions can be used, too, provided the Laplacian is restricted to the complement of its kernel. For details, see, e.g., Chapter 2 of [16].

The second example is the regularized partition function for spin 1/2 fermions on a finite sublattice Λ of \mathbb{Z}^d , $d \geq 1$. Here $\Gamma = \{0, \frac{\beta}{n}, 2\frac{\beta}{n}, \dots, (n-1)\frac{\beta}{n}\} \times \Lambda$, $I = \{1, -1\} \times \{1, -1\}$, with the first index being a charge index and the second the spin index. For $X = (c, \xi) \in I \times \Gamma$, with $\xi = (\alpha, \tau, x)$ and $X' = (c', \xi')$, with $\xi' = (\alpha', \tau', x')$,

$$(A_0)_{X,X'} = \begin{pmatrix} 0 & \tilde{A}_0(\xi, \xi') \\ -\tilde{A}_0(\xi', \xi) & 0 \end{pmatrix}_{c,c'} \quad (6)$$

where $\tilde{A}_0(\xi, \xi') = \delta_{\alpha,\alpha'} a_0(\tau - \tau', x - x')$ and the Fourier transform of a_0 is of the form $\hat{a}_0(\omega, k) = i\omega - e_0(k)$, where the Matsubara frequency ω takes values in odd multiples of π times the temperature $T = \beta^{-1}$ and e_0 is a real-valued function that has a nontrivial zero level set, the Fermi surface $S_0 = \{k : e_0(k) = 0\}$. $\mathcal{V}_0(\Psi)$ is an even element of the Grassmann algebra with $\mathcal{V}_0(0) = 0$. In the simplest case, \mathcal{V}_0 is a quartic polynomial in the fields, e.g. $\sum_{X_1, \dots, X_4} v_0(X_1, \dots, X_4) \Psi_{X_1} \dots \Psi_{X_4}$, but it can be much more general. The number n is an auxiliary quantity coming from an application of the Lie–Trotter product formula. To recover a representation of the partition function of the many–fermion model, as given by a trace over Fock space, one must take $n \rightarrow \infty$ first. For details, see Chapter 4 of [16].

2.2 Renormalization group

The renormalization group comes into play when the inverse of A_0 becomes unbounded, or even fails to exist, in the limit $|\mathcal{X}| \rightarrow \infty$, or when the inverse remains bounded, but its norm becomes nonuniform in an essential physical parameter. In the above examples, the Fourier transform diagonalizes A_0 and one can read off

that the eigenvalues go to zero in the limit of interest. In Example 1 above, taking Γ to be a cube of sidelength L , $k_i = n_i 2\pi/L$ with $n_i \in \mathbb{N}_0$; clearly, $\hat{a}_0(k)$ goes to zero for infinitely many $n = (n_1, n_2, n_3)$ in that limit, so that $\hat{C} = \hat{a}_0^{-1}$ becomes singular. Similarly, in Example 2, one gets zero eigenvalues when the temperature $T \rightarrow 0$, and when $k \in S_0$. At small positive T , $\hat{C} = \hat{a}_0^{-1}$ is of order $1/T$. In both cases, \hat{C} , and thus C itself, fails to be square integrable in the limit $T \rightarrow 0$, so that integrals over a square of a propagator (and higher powers of it) are ill-defined. This is not just mathematical pedantry but physically relevant – if in the case of the many-fermion system, \hat{C} were square integrable at $T = 0$, there would be no superconductivity, because the superconducting gap is driven by the $\log \beta$ from $\int |\hat{C}|^2$. Similarly, the singularity of \hat{a}_0^{-1} in the bosonic case leads to anomalous decay exponents of correlation functions. Hence in the following, a guideline for getting a useful RG equation in these systems is that *squares of unregularized propagators are to be avoided*.

A standard approach to obtain a renormalization group equation [10] in the above examples is as follows. Introduce a scale parameter $s \geq 0$, an s -dependent energy, say $\epsilon_s = \epsilon_0 e^{-s}$, which goes to zero in the limit $s \rightarrow \infty$. Choose a fixed decreasing C^∞ cutoff function $\chi_<$ with $\chi_<(x) = 1$ for $x \leq 1/4$ and $\chi_<(x) = 0$ for $x \geq 1$, and set $\chi_> = 1 - \chi_<$, so that $\chi_< + \chi_> = 1$ is a smooth partition of unity. Decompose the propagator into an infrared part D_s and a part supported at higher scales, C_s , via

$$\hat{C}(k) = \hat{C}_s(k) + \hat{D}_s(k) = \chi_>\left(\frac{|\hat{a}_0(k)|^2}{\epsilon_s^2}\right) \hat{C}(k) + \chi_<\left(\frac{|\hat{a}_0(k)|^2}{\epsilon_s^2}\right) \hat{C}(k). \quad (7)$$

and define the effective action

$$\mathcal{G}_s(\Phi) = -\log \int d\mu_{C_s}(\Theta) e^{-\mathcal{V}(\Theta+\Phi)}. \quad (8)$$

The partition function is then expressed in terms of the effective action as

$$P_0(\Psi) = e^{\frac{1}{2}(C\Psi, \Psi)} \int d\mu_{D_s}(\Theta) e^{-\mathcal{G}_s(\Theta+\Phi)}. \quad (9)$$

The effective action \mathcal{G}_s is a function of a fields which, when expanded in the fields, has a quadratic term. In infrared problems, this term is relevant in the RG sense. That is, the order of the singularity may get changed, producing the anomalous exponents, in the scalar field theory example, or, in the many-fermion system, the changes to the quadratic term also shift the location of the singularity of the

propagator, the Fermi surface, in Fourier space. The above scale decomposition does not take this into account, because it zooms in on the fixed Fermi surface S_0 . Therefore, with such a fixed choice of scale decomposition, counterterms are needed to avoid divergences [2]. It is possible to justify the use of counterterms by an inversion theorem [2], but it seems desirable to have a simple method for moving the quadratic terms from \mathcal{G}_s into the propagator, in a way that the shift of the singularity gets included in the flow automatically. The flow introduced in this paper allows to put self-energy terms into the propagator and hence to define a convenient adaptive scale decomposition both in continuous (and in discrete) renormalization group flows. It is such that an analogue of (9) continues to hold, but with a propagator that changes dynamically in the flow, and $\mathcal{G}(s)$ replaced by an interaction from which all, or part of, the quadratic part is removed.

Finding a scheme that implements an adjusting scale decomposition in a differential equation is not completely straightforward. On graphical grounds, one expects the full propagator to be $G = (A - \Sigma)^{-1}$, where Σ is the Dyson self-energy. Thus the first idea may be to make the ansatz

$$C(s) = A(s)^{-1} \chi_{>} \left(\frac{A(s)^* A(s)}{\epsilon_s^2} \right) \quad (10)$$

with $A(s) = A_0 - \Sigma(s)$, $\Sigma(s)$ to be determined. $C(s)$ is well-defined because $\chi_{>}$ is nonzero only on a subspace where $A_0 - \Sigma(s)$ can be inverted, and the cutoff function now adapts dynamically because it depends on $\Sigma(s)$. The propagator of the unintegrated fields must be defined as $D(s) = C - C(s)$ to ensure that $P(s, \Psi) = P_0(\Psi)$. But this has a singularity at the noninteracting Fermi surface for any s , as well as one developing on the new one, so (apart from being a very unnatural object) it does not vanish for $s \rightarrow \infty$, so that not all degrees of freedom get integrated out. Then this flow would not give useful information about the model, even if it converged for $s \rightarrow \infty$. Trying instead to keep the form of (9), but putting

$$D(s) = A(s)^{-1} \chi_{<} \left(\frac{A(s)^* A(s)}{\epsilon_s^2} \right) \quad (11)$$

leads to disaster: The derivative $\frac{\partial D(s)}{\partial s}$ contains a term $-A(s)^{-1} \frac{\partial \Sigma(s)}{\partial s} A(s)^{-1} \chi_{<}$, which has a square of the inverse, *but no infrared cutoff* because $\chi_{<}(x) = 1$ for $x \leq 1/4$. In all cases where the self-energy shifts the Fermi surface, $\frac{\partial \Sigma(s)}{\partial s} \neq 0$ on the singularity set, so that effectively, a square of the propagator appears. Because this propagator is not square integrable, this scheme leads to divergent integrals

already in the RG differential equation at fixed s . The method developed in the following sections avoids these problems.

2.3 Conditions on the adaptive flow

I now return to the general field theoretic situation of Section 2.1 and consider the generating function in the form (5), i.e.

$$P_0(\Psi) = e^{\frac{1}{2}(D_0\Psi, \Psi)} \int d\mu_{D_0}(\Theta) e^{-\mathcal{V}_0(\Theta + D_0\Psi)} \quad (12)$$

Instead of $D_0 = C = A_0^{-1}$ I make the slightly more general choice

$$D_0 = A_0^{-1}\chi_0 \quad (13)$$

where χ_0 can be chosen as an operator that cuts off very large energies, chosen such that D_0 is ζ -symmetric, and for bosons, nonnegative (but $\chi_0 = 1$ is also allowed). In applications, such an ultraviolet cutoff allows to start the flow at a scale where degrees of freedom with very high energies have already been integrated over. The flow is set up by posing restrictions on the general, s -dependent partition function

$$P(s, \Psi) = e^{K(s) + \frac{1}{2}(G(s)\Psi, \Psi)} \int d\mu_{D(s)}(\Theta) e^{-\mathcal{V}(s, \Theta + S(s)\Psi)}. \quad (14)$$

Here K , G , $\mathcal{V}(\cdot, \Psi)$, S , and D to be determined as functions of s . For fixed s , $K(s)$ is a complex number, S is an operator, G and D are ζ -symmetric operators, and $\mathcal{V}(s, \Psi)$ is a function of the fields Ψ , which, for bosons, has to be such that the integral exists (see below).

I impose the following general conditions.

$$\text{H1 } K(0) = 0, G(0) = D(0) = S(0) = D_0, \text{ and } \mathcal{V}(0, \psi) = V_0(\psi).$$

$$\text{H2 } \text{For all } s \geq 0 \text{ and all } \Psi, P(s, \Psi) = P(0, \Psi).$$

$$\text{H3 } D(s) \rightarrow 0 \text{ as } s \rightarrow \infty.$$

These conditions have the following simple interpretation.

1. $P(0, \Psi) = P_0(\Psi)$ is the generating function of the model of interest.

2. The generating function P , and hence all correlation functions, are invariant under the RG.
3. In the end, all degrees of freedom are integrated out.

The field-independent term $-K(s)$ is (up to a factor β^{-1}) equal to the free energy. The term involving $G(s)$ naturally arises once one starts adding terms to the denominator of the propagator. This is most easily seen in the case when one requires that $\mathcal{V}(s, \Psi)$ should contain no quadratic term in Ψ . Then, if the limit $s \rightarrow \infty$ exists and $D(s) \rightarrow 0$ in that limit (as required by condition 3),

$$P(\infty, \psi) = e^{K_\infty + \frac{1}{2}(G_\infty \Psi, \Psi)} e^{-\mathcal{V}_\infty(S_\infty \Psi)}. \quad (15)$$

Because $\mathcal{V}(s)$ has no quadratic term for any s , the same holds for $\mathcal{V}_\infty = \lim_{s \rightarrow \infty} \mathcal{V}(s)$. Therefore the term involving G_∞ is the only quadratic term and hence G_∞ is the full propagator. The limit exists e.g. if the temperature is above all critical temperatures [10, 21, 3] or if the Fermi surface is asymmetric under reflections [22].

Thus, the full propagator G is built up gradually in the RG flow. For finite s , (14) has the usual RG interpretation that $D(s)$ is the propagator of particles with interaction $\mathcal{V}(s)$, but now $\mathcal{V}(s)$ can be chosen not to contain any self-energy terms. Thus, even if the limit $s \rightarrow \infty$ cannot be taken because the flow is accurate only up to some s_0 (see, e.g. [11, 8]), the effective propagator and interaction can still be used as an input for the theory of the excitations with energy below ϵ_{s_0} .

The above conditions do not determine $K(s)$ and the functions $G(s)$, $\mathcal{V}(s, \Psi)$, $S(s)$, and $D(s)$ uniquely. They are just the minimal conditions one wants to impose. The freedom of choosing the various functions can be used to impose the constraint that the quadratic part of $\mathcal{V}(s)$ should vanish, or that a certain part of it be removed from $\mathcal{V}(s)$ and put into $D(s)$ instead.

The flow (9) satisfies H1–H3, but it restricts to the propagator $D(s) = \chi_s C$ with the fixed scale function χ_s of (7), which does not adjust to a shifting location of the singularity. This condition is relaxed when one drops (9) in favour of (14). In particular, one can choose the functions such that the propagator $D(s)$ incorporates all, or part of, the selfenergy corrections. In applications, this means that one can directly study the flow of the Fermi surface, the flow of superconducting gaps and the like, without having to use counterterms.

2.4 The RG differential equations

I seek only solutions to the conditions H1–H3 for which K, S, G, D and \mathcal{V} are differentiable in s . Then H1 and H2 are equivalent to the initial–value problem

$$\begin{aligned} \frac{\partial}{\partial s} P(s, \eta) &= 0 \\ K(0) = 0, \quad G(0) = D(0) = S(0) &= D_0, \quad \mathcal{V}(0, \psi) = \mathcal{V}_0(\psi), \end{aligned} \quad (16)$$

from which the RG differential equation will be obtained.

For a ζ –symmetric operator A , the Laplacian in field space is defined as

$$\Delta_A = (\delta_\Psi, A \delta_\Psi) \quad (17)$$

where $\delta_\Psi = \alpha^{-1} \frac{\partial}{\partial \Psi}$.

Theorem 2.1 *Let $s \mapsto Q(s)$ be given, with $Q(s)$ ζ –symmetric for all s and bounded for all s . Assume that there is a solution (A, χ) to the initial–value problem*

$$\dot{A}(s) = -\chi(s)Q(s), \quad A(0) = A_0, \quad \chi(0) = \chi_0, \quad (18)$$

such that $s \mapsto \chi(s)$ is differentiable and the low–energy propagator

$$D(s) = A(s)^{-1} \chi(s) \quad (19)$$

and the fluctuation propagator

$$F(s) = A(s)^{-1} \frac{\partial}{\partial s} \chi(s) \quad (20)$$

exist and are ζ –symmetric, and that $D(s)$ is nonnegative for bosons. Assume that K and \mathcal{V} satisfy the initial–value problem

$$\begin{aligned} \dot{\mathcal{V}}(s, \Psi) &= e^{\mathcal{V}(s, \Psi)} \frac{1}{2} \Delta_{F(s)} e^{-\mathcal{V}(s, \Psi)} + \frac{1}{2} (Q(s) \Psi, \Psi) + \dot{K} - \frac{\zeta}{2} \text{Tr} (DQ), \\ \mathcal{V}(0, \Psi) &= \mathcal{V}_0(\Psi), \quad K(0) = 0 \end{aligned} \quad (21)$$

(thus in particular $\mathcal{V}(s, \Psi)$ is differentiable in s and at least twice differentiable in Ψ) and the condition that $\mathcal{V}(s, 0) = 0$ for all s . Set

$$S(s) = A(s)^{-1} \chi_0 \quad (22)$$

and let G be the solution to

$$\dot{G}(s) = S(s)^T Q(s) S(s), \quad G(0) = D_0. \quad (23)$$

Then the functions K, G, D, S and \mathcal{V} solve the initial value problem (16), hence they satisfy H1 and H2.

This theorem is proven in the next subsection.

Remarks.

1. D must be ζ -symmetric, and for bosons it must be nonnegative. A way to achieve this is to restrict to ζ -symmetric $A(s)$ that commutes with its adjoint $A^*(s)$, and to take $\chi(s)$ as a function of $A^*A(s)$, so that the two commute, e.g. $\chi(s) = \chi_{<} \left(\frac{A(s)^*A(s)}{\epsilon_s^2} \right)$. The conditions on A are satisfied in the above examples and in most other interesting applications.
2. The dependence of the cutoff function $\chi(s)$ on $A(s)$ allows for an s -dependent, adaptive scale decomposition. This makes showing the existence of a solution to (18) nontrivial. This existence problem will be discussed in more detail in Section 2.9.
3. The fluctuation propagator $F(s)$ given in (20) is now indeed proportional to the derivative of the scale function, thus has an infrared cutoff. Therefore the problem mentioned after (11) is absent from this flow.
4. The equation for the full propagator G can be rewritten as

$$G(s) = A(s)^{-1} - \int_0^s dt A(t)^{-1}(1 - \chi(t))Q(t)A(t)^{-1}. \quad (24)$$

Thus in general, $G(s) \neq A(s)^{-1}$, also in the limit as $s \rightarrow \infty$. This must be so for general reasons, as will be explained in Section 2.6.

5. The strategy to satisfy H3 as well is to choose a solution for which the support of $\chi(s)$ becomes empty as $s \rightarrow \infty$, and for which the singularity of $A(s)^{-1}$ as $s \rightarrow \infty$ is such that $D(s) = A(s)^{-1}\chi(s)$ still vanishes in the limit.
6. Theorem 2.1 gives a family of solutions to the conditions H1 and H2 parametrized by Q . Moreover, it will become clear in the proof that even given a fixed Q , there are many other possibilities to choose the equation for \mathcal{V} , and as explained, there is a further freedom of choosing χ as a function of A when seeking a solution (χ, A) to (18). Thus, there seems to be an enormous freedom in solving H1–H2. However, not every choice has good analytical properties, and most solutions diverge already at a small value of s . To have a bounded solution on a maximal interval for s requires a careful

choice of $Q(s)$. For instance, in the Fermi surface problem in $d \geq 2$, Q must take the Fermi surface shift into account fully; whether it also needs to contain corrections to the Fermi velocity and the coefficient of ω depends on finer details, such as the shape of the Fermi surface. If this is done properly, the limit $s \rightarrow \infty$ of the flow exists at temperatures above the mean-field transition temperature [3], and H3 is satisfied.

7. Compared to the Polchinski equation obtained from (9) by differentiating with respect to s , which reads

$$\dot{\mathcal{G}}_s = e^{\mathcal{G}_s} \frac{1}{2} \Delta_{\dot{D}_s} e^{-\mathcal{G}_s}, \quad (25)$$

(with $D_s = C\chi_{<}$ as in (7)), there are two changes. First, there are terms of order 0 and 2 in the fields Ψ on the right hand side of (21), which can be chosen to cancel the field-independent and quadratic part arising from the first term on the right hand side, so that $\dot{\mathcal{V}}(s)$ starts with terms of order 4 in Ψ . Second, the fluctuation propagator $F(s)$ now includes selfenergy corrections from the integration up to scale s (if $Q(s)$ is chosen to absorb only selected parts of the quadratic part generated by the integration, e.g. only cancelling the Fermi surface shift, then the propagator $A(s)$, and hence $F(s)$, contain only those).

Thus (21) has a simple and natural interpretation: the propagator is allowed to change dynamically, and the selfenergy (or part of it) can be put into the denominator. Although the RHS of (25) has the same structure as the first term on the RHS of (21), the change of the quadratic part, and the ensuing replacement of \dot{C}_s by $F(s)$ leads to a major change in the behaviour of the RG equations. Some aspects will be discussed in Section 4.

2.5 Proof of Theorem 2.1

To make the proof more readable, I first give an algebraic derivation which applies both to bosons and fermions, and in a subsequent section provide the mathematical justifications of all steps.

2.5.1 Derivation of the RG equations

Define the convolution with the Gaussian measure as

$$(\mu_D * F)(\Phi) = \int d\mu_D(\Theta) F(\Theta + \Phi) \quad (26)$$

The partition function is

$$P(s, \eta) = e^{K(s) + \frac{1}{2}(G(s)\eta, \eta)} (\mu_{D(s)} * e^{-\mathcal{V}(s)}) (S(s)\eta) \quad (27)$$

If W is differentiable in s and Ψ ,

$$\frac{d}{ds} W(s, S(s)\eta) = \left[\frac{\partial W}{\partial s}(s, \Psi) + \mathcal{D}_{H(s)} W(s, \Psi) \right]_{\Psi=S(s)\eta} \quad (28)$$

with

$$H(s) = \dot{S}(s) S(s)^{-1} \text{ and } \mathcal{D}_H = (H \Psi, \delta_\Psi). \quad (29)$$

The equation $\frac{dP}{ds} = 0$ implies

$$\left(\dot{K}(s) + \frac{1}{2}(Q(s)\Psi, \Psi) + \mathcal{D}_{H(s)} + \frac{\partial}{\partial s} \right) (\mu_{D(s)} * e^{-\mathcal{V}(s)}) (\Psi) = 0 \quad (30)$$

with

$$Q(s) = (S(s)^{-1})^T \dot{G}(s) S(s)^{-1}. \quad (31)$$

Using

$$(\mu_{D(s)} * e^{-\mathcal{V}(s, \Psi)}) (\Psi) = e^{\frac{1}{2}\Delta_{D(s)}} e^{-\mathcal{V}(s, \Psi)} (\Psi) \quad (32)$$

gives

$$\left(\dot{K}(s) + \frac{1}{2}(Q(s)\Psi, \Psi) + \mathcal{D}_{H(s)} + \frac{\partial}{\partial s} \right) e^{\frac{1}{2}\Delta_{D(s)}} e^{-\mathcal{V}(s, \Psi)} = 0 \quad (33)$$

Acting with $e^{\mathcal{V}(s, \Psi)} e^{-\frac{1}{2}\Delta_{D(s)}}$ on (33) and using

$$e^{-\frac{1}{2}\Delta_D} \mathcal{D}_H e^{\frac{1}{2}\Delta_D} = \mathcal{D}_H - \frac{1}{2}\Delta_{HD+DH^T} \quad (34)$$

$$e^{-\frac{1}{2}\Delta_D} \frac{1}{2}(Q\Psi, \Psi) e^{\frac{1}{2}\Delta_D} = \frac{1}{2}(Q\Psi, \Psi) - \frac{\zeta}{2} \text{Tr}(DQ) + \frac{1}{2}\Delta_{DQD} - \mathcal{D}_{DQ} \quad (35)$$

(recall $\zeta = 1$ for bosons and $\zeta = -1$ for fermions) I get

$$\dot{V} = e^V \left(\dot{K} - \frac{\zeta}{2} \text{Tr}(DQ) + \frac{1}{2}(Q\Psi, \Psi) + \frac{1}{2}\Delta_F + \mathcal{D}_B \right) e^{-V} \quad (36)$$

with

$$F = \dot{D} - HD - DH^T + DQD, \quad B = H - DQ. \quad (37)$$

To simplify the equations, I remove the dilation term \mathcal{D}_B , i.e. choose S so that

$$H = S^{-1}\dot{S} = DQ \quad (38)$$

This implies $F = \dot{D} - DQD$, so that

$$\dot{V}(\Psi) = \dot{K} - \frac{\zeta}{2} \text{Tr}(DQ) + \frac{1}{2}(Q\Psi, \Psi) + e^{\mathcal{V}(\Psi)} \frac{1}{2} \Delta_{\dot{D}-DQD} e^{-\mathcal{V}(\Psi)} \quad (39)$$

This equation makes it natural to regard $Q(s)$ as an input, and to seek a solution for D, \mathcal{V} , and K . Once this is done, the remaining functions are obtained by integrating the equations $\dot{S} = DQ S$ and $\dot{G} = S^T Q S$.

Let D be of the form (19) with $A(s)$ and $\chi(s)$ the solution to (18). Differentiation with respect to s implies that $F(s)$ is indeed given by (20). With the choice (19), the amputation operator S can now also be determined: $\dot{S} = DQ S = -\dot{A} A^{-1} S$, with initial condition $S(0) = A(0)^{-1} \chi_0$ has the unique solution $S(s) = A(s)^{-1} \chi_0$ stated in (22).

Thus Theorem 2.1 is proven algebraically for bosons and fermions. Adapting the derivation to the case of a theory involving bosons and fermions is straightforward, provided that the interaction V_0 is an even element of the Grassmann algebra.

2.5.2 Mathematical remarks

In the above derivation, it was assumed

1. that \mathcal{V} and P are indeed differentiable in s and in Ψ
2. that (32) holds, and
3. that the commutator equations hold, when applied to $e^{-\mathcal{V}}$.

Item 1 follows from the existence of the solution to the initial value problem for K and \mathcal{V} , as stated in the hypotheses of Theorem 2.1 (and already explained there).

Concerning item 2, note that for any polynomial Π in the fields,

$$(\mu_{D(s)} * \Pi)(\Psi) = e^{\frac{1}{2} \Delta_{D(s)}} \Pi(\Psi) \quad (40)$$

was proven in [16] both for bosons and fermions, but $e^{-\mathcal{V}}$ is in general not a polynomial. Similarly, the commutator equations (34) and (35) hold when applied to polynomials in Ψ , but again, the above proof involves an application to $e^{-\mathcal{V}}$.

Consider first the case of fermions. In the present setup, the Grassmann algebra is finite-dimensional, so every function of the Grassmann variables is a polynomial. Thus $\mathcal{V}(s, \Theta)$ and $e^{\pm\mathcal{V}(s, \Theta)}$ are polynomials in Θ and $P(s, \Psi)$ is a polynomial, hence infinitely differentiable in Ψ . Thus (32) holds, and (34) and (35) hold as well. Moreover χ is differentiable in s by hypothesis of Theorem 2.1, and A is differentiable in s because it solves the differential equation (18), so D is differentiable in s as well. By (32), the convolution in (14) is a polynomial in $D(s)$ and in $\mathcal{V}(s, \cdot)$. Thus by expanding \mathcal{V} in homogeneous parts of degree $2m$ in Ψ (see also Section 2.6), one easily proves by induction on m that \mathcal{V} , and hence also P , is differentiable in s .

For bosons, $e^{-\mathcal{V}(s, \Psi)}$ is not a polynomial, but, under our assumptions, it has a fast decay for $|\Psi| \rightarrow \infty$. This decay and some analyticity could be used to justify first truncating $e^{-\mathcal{V}}$ to a polynomial and then taking a limit. A different, easier way to adapt the argument is to avoid (32), i.e. go back to (30) and use integration by parts—arguments of the type

$$\Psi \int d\mu_D(\Psi - \Theta) F(\Theta) = \int d\mu_d(\Psi - \Theta) \left(\Theta - D \frac{\partial}{\partial \Theta} \right) F(\Theta) \quad (41)$$

to bring the equation into the form

$$\int d\mu_{D(s)}(\Psi - \Theta) \mathcal{F}(s, \Theta) = 0 \quad (42)$$

where $\mathcal{F}(s, \Psi)$ is the difference between the left and right hand sides of (39). Keeping $D(s) > 0$ in the derivation and obtaining the general case of nonnegative D by a limit, an equation of this form implies that $\mathcal{F}(s, \Theta) = 0$. This yields the same differential equation as the algebraic method of the previous section and thus completes the derivation for bosons.

In conclusion, all steps in the derivation of the RG equations in the previous subsection are justified mathematically. In the case of fermions, the smoothness of $\mathcal{V}(s, \Psi)$ in s and Ψ follows from easy general arguments. For bosons, it is part of showing that the solution to the equations for \mathcal{V} and K exists.

2.6 Structure of the dynRG hierarchy

For brevity I call the RG differential equation for \mathcal{V} , K and A , χ derived above the dynamical RG or dynRG equation. Expansion of \mathcal{V} in powers of the fields gives a hierarchy of equations for the m -point functions. In this section, I assume

that \mathcal{V} is even and choose Q and K such that \mathcal{V} does not contain any quadratic and constant part, which gives the hierarchy with the simplest graphical structure. The expansion in the fields

$$\mathcal{V}(\Psi) = \sum_{m \geq 2} \mathcal{V}^{(m)}(\Psi) \quad (43)$$

with $\mathcal{V}^{(m)}$ homogeneous of degree $2m$ in Ψ , leads to the hierarchy (shown graphically for Q , $\mathcal{V}^{(2)}$ and $\mathcal{V}^{(3)}$ in Figure 1)

$$\dot{\mathcal{V}}^{(m)} = \sum_{n=2}^{m-1} \frac{1}{2} (\delta_{\Psi} \mathcal{V}^{(n)}, F \delta_{\Psi} \mathcal{V}^{(m+1-n)}) - \frac{1}{2} \Delta_F \mathcal{V}^{(m+1)}, \quad m \geq 2 \quad (44)$$

$$(Q(s)\Psi, \Psi) = \Delta_{F(s)} \mathcal{V}^{(2)}(s, \Psi) \quad (45)$$

$$\dot{K} = \frac{\zeta}{2} \text{Tr} (DQ) \quad (46)$$

By (18), (46) reads $\dot{K} = -\frac{\zeta}{2} \text{Tr} (A^{-1} \dot{A}) = -\frac{\partial}{\partial s} \frac{\zeta}{2} \text{Tr} \log A$, so it integrates to

$$K(s) = -\frac{\zeta}{2} (\text{Tr} \log A(s) - \text{Tr} \log A(0)) \quad (47)$$

For fermions, $\zeta = -1$ and $\text{Tr} \log A = \text{Pf } A$, where $\text{Pf}(A)$ denotes the Pfaffian of the antisymmetric matrix A . For covariances of the form (6), $\text{Pf}(A) = (\det \tilde{A})^2$, so that

$$e^{K(s)} = \frac{\det \tilde{A}(s)}{\det \tilde{A}(0)} \quad (48)$$

For bosons, $\zeta = 1$ and $\text{Tr} \log A = \det A$. Thus for real bosons,

$$e^{K(s)} = \left(\frac{\det A(s)}{\det A(0)} \right)^{-1/2} \quad (49)$$

For complex bosons with a hermitian covariance there is a factor 2 which squares this to $\det A(0)/\det A(s)$.

Thus the relation of the operator A to the free energy density is simple: if $A(s) \rightarrow A$ for $s \rightarrow \infty$, the free energy is (modulo a conventional factor of the inverse temperature) simply given by

$$K = -\frac{\zeta}{2} (\text{Tr} \log A - \text{Tr} \log A_0) \quad (50)$$

This also clarifies why the equation (24) for G cannot simply be $G = A^{-1}$: the exact relation between the full propagator G and the free energy involves the Luttinger–Ward functional. It is a nice feature of the dynRG flow that the relation between A and the free energy K is so simple.

Clearly, the diagrammatic structure of the hierarchy (see Figure 1) differs from the Polchinski hierarchy obtained from (25) by the absence of one–particle functions (or two–point functions) which get absorbed into A via the equation (45) for Q and (18).

The standard iterative solution of this hierarchy produces a graphical expansion. It should be noted that from a purely graphical perspective, the dynamical propagator adjustment does not make any of the functions one–irreducible. This is obvious for the six–point and higher functions, but also holds for the graphs contributing to the four–point function $\mathcal{V}^{(2)}$ and to Q : taking the tree diagram of the six–point function and joining two legs of one of the vertices to a tadpole produces a one–reducible contribution to $\mathcal{V}^{(2)}$, taking two legs of the other vertex and joining them to a tadpole gives a one–reducible contribution to Q . The point here is that these two–point insertions (the *single-scale insertions* defined in [2]) are harmless because they do not generate singularities. In the Fermi surface example, the size of such an insertion at scale s is by power counting (see [10]) of order $\epsilon_s \sim e^{-s}$, while the size of the propagators on the adjoining lines is of order $\epsilon_s^{-1} \sim e^s$. Thus the combination of two propagators and such a two–point insertion is of order $e^{-s}e^{2s} = e^s$, which is the same size as a single propagator. However, if one inserted a two–point graph that is integrated from 0 to s , its value would be $\sim \int_0^s ds' e^{-s'} = O(1)$. Then the combination of this insertion with two propagators is of size e^{2s} , hence e^s larger than it should be. It is contributions of this type that lead to a problem with unrenormalized expansions. Here, this problem does not arise because renormalization is implemented by a changing propagator. In summary, for the analytic questions, namely which terms are large and which are small, the reducible graphs contributing to $\mathcal{V}^{(2)}$ and Q are inessential.

2.7 Wick ordering and self–consistency equations

Expanding in Wick ordered polynomials provides essential simplifications in the mathematical analysis of the RG equations [10, 16]; moreover the Wick ordered scheme has been useful for calculations in two–dimensional systems [11, 23]. A distinctive feature of the Wick ordered equation is that all propagators are supported on scales below ϵ_s , so that only low–energy degrees of freedom enter the integration. In the many–fermion example, this means that all momentum inte-

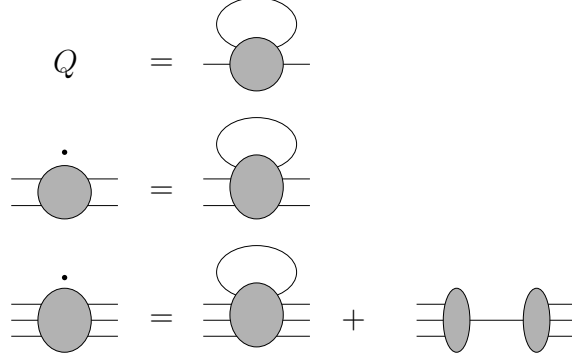


Figure 1: The first three equations of the dynRG hierarchy. The first equation determines Q , which enters the right hand side of the differential equation for the inverse propagator. The dots denote derivatives with respect to the scale parameter s . All internal lines carry a fluctuation propagator F .

grals are restricted to a neighbourhood of the Fermi surface. Therefore projections to the Fermi surface, which are needed to solve the equations numerically, are better under control than in other schemes. An further advantage is that structures of overlapping loops [2, 10, 24] become explicit in the equations and make the study of improvements to power counting much easier. These improvements are needed for showing Fermi liquid behaviour in higher dimensions and controlling two-particle interactions.

A natural (but not the only) way to fix the Wick ordering covariance is to require that tadpoles be removed. In the case of the fixed scale decomposition (7), this is easy to do : the condition to cancel tadpoles is to have $\frac{\partial}{\partial s} R(s) = -\dot{C}_s$, because then the linear term in the RGDE drops out [10]. In this case, the solution to that equation is simply $R(s) = D_s$, with the D_s of (7). In the adaptive scheme (21), the corresponding equation is

$$\frac{\partial}{\partial s} R(s) = -F(s) \quad (51)$$

which is *not* solved by $R(s) = D(s)$. The reason for this is simply that a Wick ordering that removes all tadpoles requires information about the propagator at later scales $s' > s$, which requires solving self-consistency equations. In theoretical studies, this self-consistency can be solved by fixed-point arguments. Alternatively, one can choose the Wick ordering covariance such that it leads to the above

improvements without cancelling tadpoles exactly.

The Wick ordered equation is obtained from (21)

$$\begin{aligned}\dot{\mathcal{V}}(s, \Psi) = & -\frac{1}{2}\Delta_{F(s)}\mathcal{V}(s, \Psi) + \frac{1}{2}\left(\frac{\partial\mathcal{V}(s, \Psi)}{\partial\Psi}, F(s)\frac{\partial\mathcal{V}(s, \Psi)}{\partial\Psi}\right) \\ & + \frac{1}{2}(Q(s)\Psi, \Psi) + \dot{K} - \frac{\zeta}{2}\text{Tr}(DQ),\end{aligned}\quad (52)$$

by setting

$$\mathcal{V}(s, \Psi) = e^{-\frac{1}{2}\Delta_{R(s)}}\mathcal{W}(s, \Psi). \quad (53)$$

Then

$$\dot{\mathcal{V}}(s, \Psi) = -\frac{1}{2}\Delta_{\dot{R}(s)}\mathcal{V}(s, \Psi) + e^{-\frac{1}{2}\Delta_{R(s)}}\dot{\mathcal{W}}(s, \Psi) \quad (54)$$

and the resulting equation is

$$\begin{aligned}\dot{\mathcal{W}}(s, \Psi) = & -\frac{1}{2}\Delta_{F(s)-\dot{R}(s)}\mathcal{W}(s, \Psi) \\ & + \llbracket e^{\frac{1}{2}\Delta_{R(s)}}\frac{1}{2}\Delta_{F(s)}^{(1,2)}\mathcal{W}(s, \Psi_1)\mathcal{W}(s, \Psi_2)\rrbracket \\ & + \frac{1}{2}(Q(s)\Psi, \Psi) + \dot{K}(s) - \frac{\zeta}{2}\text{Tr}[(D(s) - R(s))Q(s)].\end{aligned}\quad (55)$$

Here the usual combinatorial device [25] of introducing two copies Ψ_1 and Ψ_2 of the field has been used to rewrite the quadratic term, and $\llbracket \dots \rrbracket$ denotes evaluation at $\Psi_1 = \Psi_2 = \Psi$.

The choice of R as the solution to

$$\dot{R}(s) = F(s) \quad (56)$$

makes the linear term in the equation drop out, as in [10]. There are some further conditions on R that were automatically satisfied in the case without propagator adjustment in [10]. The support of R needs to go to zero as $s \rightarrow \infty$, and R needs to satisfy certain power counting bounds to guarantee that Wick ordering does not introduce singularities. For this reason, rewriting the solution in terms of an initial condition as

$$R(s) = R(0) + \int_0^s dt A(t)^{-1}\dot{\chi}(t) \quad (57)$$

is not useful. Instead, requiring $R(s) \rightarrow 0$ for $s \rightarrow \infty$, the solution is conveniently written as

$$R(s) = -\int_s^\infty dt A(t)^{-1}\dot{\chi}(t) \quad (58)$$

However, in contrast to the situation without adjustment of propagators (where $R(s) = D_s$), this is really a self-consistency equation because it requires knowledge about the propagator at later RG times. For general purposes, such as order-by-order arguments and power counting, it suffices to know general properties of R , and to solve for R inductively while solving the RG equations. In practical applications, one needs to use approximate solutions \tilde{R} instead of R , for which $R - \tilde{R}$ is less singular than R itself.

2.7.1 The hierarchy of Wick ordered functions

The expansion

$$\mathcal{W}(s, \Psi) = \sum_{m \geq 2} \mathcal{W}^{(m)}(s, \Psi) \quad (59)$$

with $\mathcal{W}^{(m)}$ homogeneous of order $2m$ in Ψ gives the following hierarchy. For $m \geq 2$,

$$\begin{aligned} \dot{\mathcal{W}}^{(m)}(s, \Psi) &= -\frac{1}{2} \Delta_{F-\tilde{R}} \mathcal{W}^{(m+1)}(s, \Psi) \\ &+ \sum_{\ell \geq 0} \sum_{\mu=2}^{m+\ell-1} \frac{2^{-\ell-1}}{\ell!} [(\Delta_R^{(1,2)})^\ell \Delta_F^{(1,2)} \mathcal{W}^{(\mu)}(s, \Psi_1) \mathcal{W}^{(m+1+\ell-\mu)}(s, \Psi_2)] \end{aligned} \quad (60)$$

for $m = 1$

$$\begin{aligned} \frac{1}{2} (Q\Psi, \Psi) &= -\frac{1}{2} \Delta_{F-\tilde{R}} \mathcal{W}^{(2)}(s, \Psi) \\ &+ \sum_{\ell \geq 2} \sum_{\mu=2}^{\ell} \frac{2^{-\ell}}{\ell!} [(\Delta_R^{(1,2)})^\ell \Delta_F^{(1,2)} \mathcal{W}^{(\mu)}(s, \Psi_1) \mathcal{W}^{(2+\ell-\mu)}(s, \Psi_2)] \end{aligned} \quad (61)$$

and for $m = 0$

$$\begin{aligned} \dot{K} &= \frac{\zeta}{2} \text{Tr} [(D - R)Q] \\ &+ \sum_{\ell \geq 3} \sum_{\mu=2}^{\ell-1} \frac{2^{-\ell-1}}{\ell!} [(\Delta_R^{(1,2)})^\ell \Delta_F^{(1,2)} \mathcal{W}^{(\mu)}(s, \Psi_1) \mathcal{W}^{(1+\ell-\mu)}(s, \Psi_2)] \end{aligned} \quad (62)$$

This hierarchy is shown graphically in Figure 2. The number of loops of the corresponding graphs is ℓ . As is explicit in the above equation, the contributions quadratic in \mathcal{W} start at two-loop for \dot{Q} and at three-loop for \dot{K} . The choice

$$\begin{aligned}
Q &= \text{diagram 1} + \text{diagram 2} + \text{diagram 3} \\
\dot{\text{diagram 4}} &= \text{diagram 5} + \text{diagram 6} + \text{diagram 7} \\
\dot{\text{diagram 8}} &= \text{diagram 9} + \text{diagram 10} + \text{diagram 11}
\end{aligned}$$

Figure 2: The first three equations of the Wick ordered dynRG hierarchy, with the choice $\dot{R} = F$ and the truncation that $W^{(m)} = 0$ for $m \geq 4$. In the truncation where also $W^{(3)} = 0$, only the first of the three terms remains on the right hand side in each equation.

$F = \dot{R}$ removes the term that is linear in \mathcal{W} . A graphical analysis of the iteration implies that with this choice, there are no one-reducible contributions to $\mathcal{W}^{(2)}$ and to Q . Moreover, if the one-loop term is dropped from the eq. for the four-point function, then all graphs contributing to $\mathcal{W}^{(2)}$ and Q are 2PI.

2.8 Ladder summations and symmetry breaking

If the Wick ordering covariance R is chosen as the solution to $\dot{R} = F$ that vanishes for $s \rightarrow \infty$, as above, and if the hierarchy is truncated by setting $\mathcal{W}^{(m)} = 0$ for all $m \geq 3$, the resulting equation for $\mathcal{W}^{(2)}$ becomes

$$\begin{aligned}
\dot{\mathcal{W}}^{(2)}(s, \Psi) &= \frac{1}{4} \llbracket \Delta_{R(s)}^{(1,2)} \Delta_{F(s)}^{(1,2)} \mathcal{W}^{(2)}(s, \Psi_1) \mathcal{W}^{(2)}(s, \Psi_2) \rrbracket \\
&= \frac{1}{2} \llbracket \left(\frac{\partial}{\partial s} [\Delta_{R(s)}^{(1,2)}]^2 \right) \mathcal{W}^{(2)}(s, \Psi_1) \mathcal{W}^{(2)}(s, \Psi_2) \rrbracket \quad (63)
\end{aligned}$$

The structure of this equation is analogous to that obtained by a similar truncation from the 1PI hierarchy *after the 1PI hierarchy has been rearranged in Katanin's [26] way*: Eq. (25)) in [27] is to quadratic order in the interaction of the same form as (63). It is also analogous to equations (4.176) – (4.178) in [16] because there a product of propagators $D_s \dot{D}_s$ appears in the flow equation as well; however, there, D_s does not contain self-energy corrections). Note that in contrast to the 1PI case,

the structure of the scale derivative of a square of the Laplacian (which is essential for the solution of the selfconsistency) appears in the Wick ordered dynRG in a natural way; no rearrangement of the hierarchy is necessary.

It is thus clear that, when restricted to a particular channel, the equation is easily solved by a geometric series in analogy to the procedure in Section 4.5.4. of [16]. In the present framework, restricting to a channel means that (possibly after a change of variables), the equation for the coefficient function $W_{X,Y,X',Y'}$ of $\mathcal{W}^{(2)}$ has the structure

$$\dot{W}_{X,Y,X',Y'}(s) = \sum_{U,V,U',V'} W_{X,Y,U,V}(s) \dot{\Pi}_{U,V,U',V'}(s) W_{U',V',X',Y'}(s) \quad (64)$$

which has the structure $\dot{W} = W \dot{\Pi} W$ with the product defined in the way evident from (64), i.e. regarding pairs (U, V) as matrix indices. Because (63) contains a derivative of the square of the Laplacian, $\dot{\Pi}$ is indeed explicitly given as the derivative of a matrix Π . The solution of (64) is

$$W(s) = W(0) \left(1 - [\Pi(s) - \Pi(0)] W(0) \right)^{-1}. \quad (65)$$

That integrating the RG equation leads to such a resummation was, together with the self-energy equation, crucial for accessing the symmetry-broken regime with a fermionic flow in [27]. The above structure suggests that this can also be done using the dynamic Wick ordered hierarchy developed here.

In this example, one resums in the particle-particle channel, so $\Pi(s)$ is simply

$$\Pi_{U,V,U',V'}(s) = R(s)_{U,V} R(s)_{U',V'} \quad (66)$$

where $R(s)$ denotes the non-charge-invariant propagator with $A(s)$ given by the matrix in (37), (38) of [27], multiplied by the infrared cutoff function. However, there is a crucial difference to the 1PI scheme, which I now discuss at the example of the flow of superconductor done in [27]. By definition, the Wick ordering propagators $R(s)$ are supported at small scales below $\epsilon_0 e^{-s}$, so $\lim_{s \rightarrow \infty} R(s) = 0$. Thus $\lim_{s \rightarrow \infty} \Pi(s) = 0$, and (65) gives for $s \rightarrow \infty$

$$W_\infty = W(0) [1 + \Pi(0)W(0)]^{-1}. \quad (67)$$

That is, the solution of the channel RG equation (65) does not lead to the final, but the *initial* value of the propagator in the solution of the four-point function. Hence, in the solution of the BCS model given by this truncation, it is the initial

gap that appears in the solution for the four-point function. This seeming contradiction (when viewed from the perspective of [27], where the initial gap is put in as a symmetry-breaking parameter that is subsequently sent to zero) is resolved when one remembers that in the Wick ordering scheme one also has to Wick order the initial interaction. The requirement that the initial interaction is Wick ordered *but has no quadratic part in the fields* leads to the Hartree–Fock self-consistency equation. In the reduced BCS model, this equation yields the exact solution for the superconducting gap. That is, in the Wick ordered dynRG, the initial Wick ordering of the bare interaction already gives the exact solution of the model, so the gap is constant and equal to the initial one. The flow of $\mathcal{W}^{(2)}$ then gives the exact vertex. The two-loop contributions to Q , as well as the higher contributions to $\mathcal{W}^{(2)}$, all vanish in the thermodynamic limit, as explained in the Appendix of [27].

2.9 Relation to discrete RG flows

In this section, I discuss a discrete analogue of the flow differential equation, and show that the differential flow with a dynamically changing propagator can be obtained as a limit of this discrete equation. The successive changes of the propagator in the discrete scheme take the form of a continued fraction expansion for operators.

This approach gives a specific way to check the existence of the solution to the equation (18), which is fundamental for the flow developed here. A full mathematical proof of existence is rather nontrivial, but the following gives a clear prescription how to proceed and what to check. Discrete flows are interesting by themselves since they are used in the mathematical studies in this field.

2.9.1 Adaptive iteration

Consider again the effective action \mathcal{G} for a system of fermions with propagator D and interaction \mathcal{V} , defined by

$$e^{\mathcal{G}(D_0, \mathcal{V}_0, \Phi)} = \int d\mu_{D_0}(\Psi) e^{\mathcal{V}_0(\Psi + \Phi)}. \quad (68)$$

\mathcal{G} is of the form

$$\mathcal{G}(F, V, \Psi) = \mathcal{K}(F, V) + \frac{1}{2}(\mathcal{Q}(F, V)\Psi, \Psi) + \mathcal{I}^{(\geq 4)}(F, V)(\Psi), \quad (69)$$

where \mathcal{K} is independent of Ψ , \mathcal{Q} defines a quadratic form that gives the quadratic term of G in Ψ , and $\mathcal{I}^{(\geq 4)}$ contains all the higher powers of the fields.

By successively splitting the covariance in two pieces, integrating over fluctuations, and shifting quadratic parts into the measure, I will now recursively construct a sequence of propagators and interactions $(D_n, \mathcal{V}_n)_{n \geq 1}$ with the property that the quadratic part of \mathcal{V}_n vanishes. Assume that \mathcal{V}_0 has no constant and quadratic parts. Split $D_0 = E_0 + F_0$, then

$$e^{\mathcal{G}(D_0, \mathcal{V}_0, \Phi)} = \int d\mu_{E_0}(\Psi_1) e^{\mathcal{G}(F_0, \mathcal{V}_0, \Psi + \Phi)}. \quad (70)$$

By (69), $\mathcal{G}(F_0, \mathcal{V}_0)$ splits into a constant term $K_0 = \mathcal{K}(F_0, \mathcal{V}_0)$, a quadratic term $Q_0 = \mathcal{Q}(F_0, \mathcal{V}_0)$, and a term $\mathcal{V}_1 = \mathcal{I}^{(\geq 4)}(\mathcal{G}(F_0, \mathcal{V}_0))$. The Gaussian measure is $d\mu_{E_0}(\Psi) = \mathcal{N}(E_0) e^{-\frac{1}{2}(\Psi, E_0^{-1} \Psi)} D\Psi$, with the normalization factor \mathcal{N} for the Gaussian measure being a power of determinant or Pfaffian of E_0 , so the quadratic term given by Q_0 can be shifted into the measure, and get

$$d\mu_{E_0}(\Psi) e^{\frac{1}{2}(\Psi, Q_0 \Psi)} = \mathcal{N}(1 - E_0 Q_0) d\mu_{D_1}(\Psi) \quad (71)$$

with the propagator

$$D_1^{-1} = E_0^{-1} - Q_0, \quad (72)$$

which now includes the corrections to the selfenergy given by Q_0 . Note that Q_0 is not the proper selfenergy Σ_0 but instead the connected amputated two-point function. The relation to the selfenergy Σ_0 coming from the integration over Ψ_0 is

$$Q_0 = (1 - \Sigma_0 F_0)^{-1} \Sigma_0 \quad (73)$$

If one takes the limit of a strict support condition for F_0 , $Q_0 = \Sigma_0$ outside the support of F_0 , i.e. at low energy scales.

In writing (72), I have assumed that the RHS is invertible. This condition will be further discussed below. An obvious variant of this formula holds if one decides to put only part of Q_0 (e.g. the part that shifts the Fermi surface) into the propagator.

In the convolution integral on the right hand side of (70), $\Phi + \Psi$ appears instead of Ψ , so that

$$e^{-G(D_0, \mathcal{V}_0, \Phi)} = e^{\tilde{K} + \frac{1}{2}(\Phi, Q_0 \Phi)} \int d\mu_{D_1}(\Psi) e^{(\Psi, Q_0 \Phi) + \mathcal{V}_1(\Psi + \Phi)} \quad (74)$$

with $\tilde{K}_0 = K + \log \mathcal{N}(1 - E_0 Q_0)$. Completing the square in the Gaussian measure and using the identity $1 + D_1 Q_0 = D_1 E_0^{-1}$, gives

$$\mathcal{G}(E_0 + F_0, \mathcal{V}_0, \Phi) = \tilde{K}_0 + \frac{1}{2}(\Phi, (Q_0 + Q_0 D_1 Q_0) \Phi) + \mathcal{G}(D_1, \mathcal{V}_1, D_1 E_0^{-1} \Phi). \quad (75)$$

Iteration of this identity by splitting $D_1 = E_1 + F_1$ and proceeding as above gives

$$\begin{aligned} \mathcal{G}(D_0, \mathcal{V}_0, \Phi) &= \sum_{l=0}^{n-1} \tilde{K}_l + \frac{1}{2} \sum_{l=0}^{n-1} (S_l \Phi, Q_l (1 + D_{l+1} Q_l) S_l \Phi) \\ &+ \mathcal{G}(D_n, \mathcal{V}_n, S_n \Phi) \end{aligned} \quad (76)$$

with recursively determined propagators D_n , interactions \mathcal{V}_n and amputation operators

$$S_n = \prod_{k=0}^{n-1} D_{k+1} E_k^{-1}. \quad (77)$$

Consequently,

$$\log P(D_0, V_0, \eta) = K_n + \frac{1}{2}(G_n \eta, \eta) + G(D_n, V_n, S_n D_0 \eta) \quad (78)$$

with

$$G_n = D_0 - \sum_{l=0}^{n-1} (S_l D_0)^T Q_l (1 + D_{l+1} Q_l) S_l D_0. \quad (79)$$

If the splitting in E_n and F_n is chosen such that $D_n \rightarrow 0$ as $n \rightarrow \infty$, and if K_n , \mathcal{V}_n , S_n , and G_n converge to limits K , \mathcal{V} , S , and G in that limit,

$$\log P(D_0, \mathcal{V}_0, \eta) = K + \frac{1}{2}(G \eta, \eta) + \mathcal{V}(S \eta). \quad (80)$$

If at each iteration step the full Q_n was put into the measure to define D_{n+1} , G is the full propagator of the model, and \mathcal{V} generates the connected m -point functions of the model with $m \geq 6$, with amputation given by S . The interpretation at a finite n is similar, except that there still remains an effective interaction for the degrees of freedom that have not yet been integrated over.

In the following, I discuss two specific prescriptions for the splitting, which cover most of the applications, and derive the corresponding formulas for S_n , D_n , and G_n .

2.9.2 Difference cutoff

Assume that the original covariance D_0 is given in terms of an invertible A_0 as $D = A_0^{-1}$, and that A_0 commutes with its adjoint A_0^* . The operator $|A_0|^2 = A_0^* A_0$ is hermitian and positive and therefore has a spectrum that is a subset of $\mathbb{R}_0^+ = \{x \in \mathbb{R} : x \geq 0\}$. For $n \geq 1$, let $\chi_n + \tilde{\chi}_n = 1$ be a C^∞ -partition of unity on \mathbb{R}_0^+ , where $\tilde{\chi}_n$ is decreasing and χ_n is increasing and both are strictly positive functions. We also require that

$$\frac{\tilde{\chi}_{n+1}(x)}{\tilde{\chi}_n(x)} \leq 1, \quad (81)$$

the idea being that the $\tilde{\chi}_n$ provide lower and lower cutoffs as n increases. A possible choice is $\tilde{\chi}_n(x) = (1 + e^{\gamma_n(x-1)})^{-1}$ and $\chi_n = 1 - \tilde{\chi}_n$, but the details of this choice do not matter for the moment. The operator $\tilde{\chi}_n(A^* A)$, defined via the spectral representation of $|A|^2 = A^* A$, is a positive operator that cuts off the parts of the spectrum outside the support of $\tilde{\chi}_n$. It is not a projection because we chose a continuous partition of unity instead of a step function. With this choice of $\tilde{\chi}$ as pointwise strictly positive, $\tilde{\chi}_n(|A|^2)$ is even invertible.

Given A_0 and the interaction \mathcal{V}_0 , the sequence is constructed as follows. Set

$$E_0 = A_0^{-1} \tilde{\chi}_1(|A_0|^2). \quad (82)$$

Then $F_0 = D_0 - E_0 = A_0^{-1} \chi_1(|A_0|^2)$. Calculate $Q_0 = \mathcal{Q}(F_0, \mathcal{V}_0)$ and $\mathcal{V}_1 = \mathcal{I}^{(\geq 4)}(F_0, \mathcal{V}_0)$, and define

$$A_1 = A_0 - \tilde{\chi}_1(|A_0|^2) Q_0. \quad (83)$$

If A_1 is invertible, set $D_1 = A_1^{-1} \tilde{\chi}_1(|A_0|^2)$. Then $D_1^{-1} = E_0^{-1} - Q_0$. In general, for $n \geq 1$, and given \mathcal{V}_n and an invertible A_n , set

$$D_n = A_n^{-1} \tilde{\chi}_n(|A_{n-1}|^2), \quad E_n = A_n^{-1} \tilde{\chi}_{n+1}(|A_n|^2), \quad (84)$$

and $F_n = D_n - E_n$. Determine $Q_n = \mathcal{Q}(F_n, \mathcal{V}_n)$ and $\mathcal{V}_{n+1} = \mathcal{I}^{(\geq 4)}(F_n, \mathcal{V}_n)$, and set

$$A_{n+1} = A_n - \tilde{\chi}_{n+1}(|A_n|^2) Q_n. \quad (85)$$

If A_{n+1} is invertible, the iteration can be continued; otherwise, the iteration stops at n . The thus defined D_n and E_n satisfy

$$D_n^{-1} = E_{n-1}^{-1} - Q_{n-1}, \quad D_{n+1} E_n^{-1} = A_{n+1}^{-1} A_n, \quad (86)$$

so that

$$\prod_{n=0}^{N-1} D_{n+1} E_n^{-1} = A_N^{-1} A_0. \quad (87)$$

The condition that A_n is invertible has to be checked in every step because it is, in general, nontrivial, and its failure can mean that there is some instability. We discuss this in more detail in the application to the Fermi surface deformation below.

There is another, less obvious, property of this scheme: in the fluctuation propagator

$$F_n = A_n^{-1} \left(\tilde{\chi}_n(|A_{n-1}|^2) - \tilde{\chi}_{n+1}(|A_n|^2) \right), \quad (88)$$

the difference of cutoff functions does not necessarily give a positive operator. In a bosonic theory, this would already make the fluctuation integral ill-defined. For fermions, lack of positivity is not a problem for the definition of the fluctuation integral, but numerically, positivity is important. Roughly speaking, the difference of cutoff functions becomes negative if the change of the propagator is so strong that one starts integrating backwards.

One can modify the cutoff function by replacing A by PA , where P is some projection, whenever it appears as the argument of a cutoff function. For instance, in the fermion models, one can keep the cutoff function independent of the Matsubara frequency variables.

2.9.3 Positive cutoff function

The following alternative scheme is more straightforward but does not admit a continuum limit if one wants to keep a continuous partition of unity.

Given a single partition of unity $\chi + \tilde{\chi} = 1$, with $\tilde{\chi}$ decreasing, and a strictly decreasing sequence of positive numbers $(\epsilon_n)_{n \geq 1}$, let $\tilde{\chi}_1 = \tilde{\chi} \left(\frac{|A_0|^2}{\epsilon_1^2} \right)$ and $\chi_1 = 1 - \tilde{\chi}_1$. Set

$$E_0 = A_0^{-1} \tilde{\chi}_1, \quad F_0 = A_0^{-1} \chi_1. \quad (89)$$

Then $E_0 + F_0 = D_0$, and both have positive cutoff operators. Again, calculate $Q_0 = \mathcal{Q}(F_0, \mathcal{V}_0)$ and $\mathcal{V}_1 = \mathcal{I}^{(\geq 4)}(F_0, \mathcal{V}_0)$, and set $A_1 = A_0 - \tilde{\chi}_1 Q_0$. If A_1 is invertible, set $D_1 = A_1^{-1}$.

We now proceed in exactly the same way, by splitting $D_1 = D_1(\tilde{\chi} + \chi) = E_1 + F_1$ and so on. This builds up a product of cutoff functions, which remains positive because each factor is positive. More precisely, for $n \geq 1$, and given

A_0, \dots, A_n , let

$$\tilde{\chi}_n = \tilde{\chi}_{n-1} \tilde{\chi} \left(\frac{|A_{n-1}|^2}{\epsilon_n^2} \right) = \prod_{k=1}^n \tilde{\chi} \left(\frac{|A_{k-1}|^2}{\epsilon_k^2} \right) \quad (90)$$

and set

$$D_n = A_n^{-1} \tilde{\chi}_n, \quad E_n = A_n^{-1} \tilde{\chi}_{n+1}. \quad (91)$$

Then both E_n and

$$F_n = D_n - E_n = A_n^{-1} \tilde{\chi}_n \chi \left(\frac{|A_n|^2}{\epsilon_{n+1}^2} \right) \quad (92)$$

have a positive cutoff factor. After calculation of Q_n , set

$$A_{n+1} = A_n - \tilde{\chi}_{n+1} Q_n \quad (93)$$

and, if A_{n+1} is invertible (which is the case if Q_n is small enough), $D_{n+1} = A_{n+1}^{-1} \tilde{\chi}_{n+1}$. Again, this sequence satisfies (86).

2.9.4 The summed iteration

Both of the above schemes can be described as follows. We have a sequence of invertible operators A_n and positive operators $\tilde{\chi}_n$, such that $A_0 = A$, $\tilde{\chi}_0 = 1$, and $D_n = A_n^{-1} \tilde{\chi}_n$, $E_n = A_n^{-1} \tilde{\chi}_{n+1}$, and $A_{n+1} = A_n - \tilde{\chi}_{n+1} Q_n$, with Q_n given by the fluctuation integral with propagator $F_n = D_n - E_n$ and interaction \mathcal{V}_n . Iteration gives

$$\begin{aligned} \mathcal{G}(D_0, \mathcal{V}_0, \Phi) &= \sum_{l=0}^{n-1} \tilde{K}_l + \frac{1}{2} \sum_{l=0}^{n-1} (A_l^{-1} A_0 \Phi, (Q_l + Q_l D_{l+1} Q_l) A_l^{-1} A_0 \Phi) \\ &+ \mathcal{G}(D_n, \mathcal{V}_n, A_n^{-1} A_0 \Phi). \end{aligned} \quad (94)$$

This equation has an easy interpretation: the \mathcal{G} on the RHS contains a fluctuation integral over fields with propagator D_n and interaction \mathcal{V}_n , and the external field is amputated with A_n instead of A_0 , as it should be, since D_n is A_n^{-1} times a cutoff function. The constant and quadratic terms are the contribution to the free energy density and the full propagator.

The generating functional is thus given by

$$\log P(D_0, \mathcal{V}_0, \eta) = \bar{K}_n + \frac{1}{2} (G_n \eta, \eta) + \mathcal{G}(D_n, \mathcal{V}_n, A_n^{-1} \eta) \quad (95)$$

where \bar{K}_n is the field-independent term and

$$G_n = D_0 - \sum_{l=0}^{n-1} A_l^{-1} (Q_l + Q_l D_{l+1} Q_l) A_l^{-1} \quad (96)$$

is the contribution to the full propagator up to step n in the RG iteration.

2.9.5 Limit of a differential equation

I now use the "difference cutoff" scheme of Section 2.9.2 to obtain a continuum limit of the RG sequence. Let $\epsilon_0 > 0$ be an initial energy scale, let $\epsilon_n = \epsilon_0 - n\varepsilon$ and choose $\tilde{\chi}_n(x) = \tilde{\chi}(\frac{x}{\epsilon_n^2})$ where $\tilde{\chi} + \chi = 1$ is a fixed partition of unity on \mathbb{R}_0^+ with strictly positive functions $\tilde{\chi}$ and χ . Since there is a one-to-one relation between n and ϵ_n and we want to take a continuum limit, we now label the sequence by ϵ_n instead of n , and denote $A_n = A_{\epsilon_n}$, $\mathcal{V}_n = \mathcal{V}_{\epsilon_n}$, etc. Since in the limit $\varepsilon \rightarrow 0$ nothing is integrated over, Q_{ϵ_n} is of order ε :

$$Q_{\epsilon_n} = \varepsilon Q'_{\epsilon_n}. \quad (97)$$

Because $\epsilon_{n\pm 1} = \epsilon_n \mp \varepsilon$, the fluctuation propagator at $\epsilon = \epsilon_n$ is, by (88),

$$F_\epsilon = A_\epsilon^{-1} \left(\tilde{\chi} \left(\frac{|A_{\epsilon+\varepsilon}|^2}{\epsilon^2} \right) - \tilde{\chi} \left(\frac{|A_\epsilon|^2}{(\epsilon - \varepsilon)^2} \right) \right) \quad (98)$$

Thus, up to terms that vanish as $\varepsilon \rightarrow 0$, $F'_\epsilon = \frac{1}{\varepsilon} F_\epsilon$ is given by

$$F'_\epsilon = A_\epsilon^{-1} \frac{\partial}{\partial \epsilon} \tilde{\chi}' \left(\frac{|A_\epsilon|^2}{\epsilon^2} \right) \quad (99)$$

Note that the ϵ -derivative acts only on the cutoff function, but not on the inverse in front. The differential equation for Q_ϵ and the other functions is obtained by doing the fluctuation integral to $O(\varepsilon)$. Using (32), recalling that $F_n = F'_{\epsilon_n} \varepsilon$, and dropping terms of order ε^2 , I get

$$\begin{aligned} \mathcal{G}(F_n, \mathcal{V}_n, \Phi) &= \log((1 + \Delta_{F_{\epsilon_n}}) e^{\mathcal{V}_n(\Phi)}) \\ &= \log(e^{\mathcal{V}_n(\Phi)} (1 + e^{-\mathcal{V}_n(\Phi)} \Delta_{F_{\epsilon_n}} e^{\mathcal{V}_n(\Phi)})) \\ &= \mathcal{V}_n(\Phi) + e^{-\mathcal{V}_n(\Phi)} \Delta_{F_{\epsilon_n}} e^{\mathcal{V}_n(\Phi)}. \end{aligned} \quad (100)$$

Thus

$$\mathcal{G}(\varepsilon F'_\epsilon, \mathcal{V}_\epsilon, \Phi) = \mathcal{V}_\epsilon(\Phi) + \varepsilon \left[\Delta_{F'_\epsilon} \mathcal{V}_\epsilon(\Phi) + \frac{1}{2} \left(\frac{\delta \mathcal{V}_\epsilon}{\delta \Phi}, F'_\epsilon \frac{\delta \mathcal{V}_\epsilon}{\delta \Phi} \right) \right] \quad (101)$$

hence the increment is simply given by the right hand side of Polchinski's equation. The main difference here is, however, that \mathcal{V}_ϵ has no quadratic part. Writing $\mathcal{V}_\epsilon = \mathcal{V}_\epsilon^{(4)} + \mathcal{V}_\epsilon^{(\geq 6)}$, extracting the quadratic part of \mathcal{G} , and taking the limit $\epsilon \rightarrow 0$, gives the differential flow equation as follows. The change of the inverse propagator is given by

$$\frac{\partial}{\partial \epsilon} A_\epsilon = Q'_\epsilon \tilde{\chi} \left(\frac{|A_\epsilon|^2}{\epsilon^2} \right) \quad (102)$$

where Q'_ϵ is determined by

$$\frac{1}{2}(\Phi, Q'_\epsilon \Phi) = \Delta_{F'_\epsilon} \mathcal{V}_\epsilon^{(4)}(\Phi) \quad (103)$$

The interaction part obeys

$$\frac{\partial}{\partial \epsilon} \mathcal{V}_\epsilon(\Phi) = \Delta_{F'_\epsilon} \mathcal{V}_\epsilon^{(\geq 6)}(\Phi) + \frac{1}{2}(\delta_\Phi \mathcal{V}_\epsilon, F'_\epsilon \delta_\Phi \mathcal{V}_\epsilon) \quad (104)$$

These are the equations obtained directly in the continuum in Section 2.4 (with the change of variables $\epsilon \rightarrow s = \log \epsilon_0 / \epsilon$).

2.9.6 Wick ordering in the discrete scheme

The discrete scheme also provides insights about the choice of Wick ordering in the limit of a differential equation, and it is of independent interest. The point where things start to differ from the non-Wick-ordered scheme is the definition of D_1 . Instead of (72), D_1 is now defined such that the integral in (74) is rewritten in terms of a Wick ordered interaction. More precisely, first set $\Phi = 0$ in (74) and require that

$$\int d\mu_{(E_0^{-1} - Q_0)^{-1}}(\Psi) e^{\mathcal{V}_1(\Psi)} = e^{\tilde{K}_1} \int d\mu_{D_1}(\Psi) e^{\Omega_{D_1}(\mathcal{W}_1)(\Psi)} \quad (105)$$

where $\Omega_D(F) = e^{-\frac{1}{2}\Delta_D F}$ denotes Wick ordering of F , as in [10], and

$$\mathcal{P}^{(\leq 2)} \mathcal{W}_1 = 0, \quad (106)$$

i.e. $\mathcal{W}_1(\Psi)$ does not have constant or quadratic parts in Ψ . This last condition leads to a self-consistency equation for D_1 whose solution differs from (72). By definition of the Gaussian measure, this means that

$$\begin{aligned} & \log \mathcal{N}((E_0^{-1} - Q_0)^{-1}) - \frac{1}{2}((E_0^{-1} - Q_0)\Psi, \Psi) + \mathcal{V}_1(\Psi) \\ &= \log \mathcal{N}(D_1) + \tilde{K}_1 - \frac{1}{2}(D_1^{-1}\Psi, \Psi) + e^{-\frac{1}{2}\Delta_{D_1}} \mathcal{W}_1(\Psi) \end{aligned} \quad (107)$$

\mathcal{W}_1 is obtained by using $1 = \mathcal{P}^{(\leq 2)} + \mathcal{P}^{(\geq 4)}$ to rewrite

$$\mathcal{V}_1 = e^{-\frac{1}{2}\Delta_{D_1}} (\mathcal{P}^{(\leq 2)} + \mathcal{P}^{(\geq 4)}) e^{\frac{1}{2}\Delta_{D_1}} \mathcal{V}_1. \quad (108)$$

Since $\mathcal{P}^{(\geq 4)}$ projects out all terms constant or quadratic in the fields, this gives

$$\begin{aligned} \mathcal{W}_1 &= \mathcal{P}^{(\geq 4)} e^{\frac{1}{2}\Delta_{D_1}} \mathcal{V}_1 \\ \tilde{K}_1 &= \mathcal{P}^{(0)} e^{\frac{1}{2}\Delta_{D_1}} \mathcal{V}_1 - \frac{1}{2}\Delta_{D_1} \mathcal{P}^{(2)} e^{\frac{1}{2}\Delta_{D_1}} \mathcal{V}_1 \end{aligned} \quad (109)$$

and

$$D_1^{-1} = E_0^{-1} - Q_0 + \mathbb{P}^{(2)} e^{\frac{1}{2}\Delta_{D_1}} \mathcal{V}_1 \quad (110)$$

where for a function of the fields $F(\phi)$, $A = \mathbb{P}^{(2)} F$ is the operator in the quadratic form given by $\mathcal{P}^{(2)} F$, i.e. $\mathcal{P}^{(2)} F(\Phi) = \frac{1}{2}(A \Phi, \Phi)$. Equation (110) is the self-consistency equation for D_1 . It takes the form of a generalized Hartree–Fock–type equation because the action of $e^{\frac{1}{2}\Delta_{D_1}}$ is to create self-contractions of the vertex given by \mathcal{V}_1 with a D_1 propagator, and this feeds into D_1^{-1} in the way that is standard in self-consistency equations of mean-field type. Now that \mathcal{W}_1 and D_1 are given, one obtains the rewriting of the integral in (74) for $\Phi \neq 0$ by standard shifting of Gaussians and redefinitions of the amputation term that acts on the external field Φ .

3 Fermi Surface Flows

In this section, I briefly discuss the Fermi surface flow in $d \geq 2$ in the scheme without Wick ordering. A discrete Fermi surface flow with Wick ordering is defined and controlled mathematically for $d = 2$ in [3]. This is one of the examples that motivated the method. Although different methods, such as the T -flow of [17] have been used to calculate RG flows, and other regularizations, such as a cutoff on the frequencies only, are possible, the momentum space flow remains the only one where precise mathematical estimates for power counting of all Green functions, not just the first few truncations, have been given [2].

Let the covariance and interaction be those of the standard many-fermion system with short-range interaction \mathcal{V}_0 , as described in the Introduction, with $n \rightarrow \infty$. (for more details, see [16], Chapter 4). In Fourier space, the covariance is a multiplication operator, namely the inverse of

$$A_0(q_0, q) = iq_0 - e_0(q), \quad (111)$$

times the unit matrix in spin space. Here e_0 is the dispersion relation for free particles, given by the hopping amplitudes of the model. By standard (formal) diagrammatics, the full propagator is the inverse of $iq_0 - e_0(q) - \sigma(q_0, q)$, with σ the fermion self-energy. One natural choice for would then be to take A as the full propagator, with σ understood to depend on s (but without a cutoff function). In the following, I shall discuss another choice, where only the frequency zero part of the selfenergy is put into the denominator. This then implies that V still has a quadratic part, but one which vanishes at zero frequency. In cases where the field renormalization is a marginally relevant parameter, one can also put in the linear part of σ in q_0 . Thus the freedom in choosing Q and A allows to adapt to the situation at hand, while keeping the denominator as simple as possible.

So, let

$$A(q_0, q) = iq_0 - e(s, q). \quad (112)$$

In terms of the scale-dependent self-energy $\sigma_s(q_0, q)$, $e(s, q) = e_0(q) + \sigma_s(0, q)$, so that the difference

$$\tilde{\sigma}_s(q_0, q) = \sigma_s(q_0, q) - \sigma_s(0, q) \quad (113)$$

vanishes at $q_0 = 0$. Let $\tilde{\chi}$ be a decreasing cutoff function with $\tilde{\chi}(x) = 1$ for $x \leq 1/2$ and $\tilde{\chi}(x) = 0$ for $x \geq 1$. Set

$$\chi(s, k) = \tilde{\chi} \left(\frac{e(s, k)^2}{\epsilon_s^2} \right). \quad (114)$$

(another choice would be to use a cutoff on the frequencies as well: $\chi(s, k_0, k) = \tilde{\chi} \left(\frac{k_0^2 + e(s, k)^2}{\epsilon_s^2} \right)$).

With this choice, $\mathcal{V}(s)$ still has a quadratic part, which is of the form

$$\int \bar{d}k \bar{\psi}(k) v_1(s, k) \psi(k) \quad (115)$$

with

$$v_1(s, (0, k)) = 0 \quad (116)$$

(here $k = (k_0, k)$, and $\int \bar{d}k = \beta^{-1} \sum_{k_0 \in M_f} \int \frac{dk}{(2\pi)^d}$ with M_f the set of fermionic Matsubara frequencies). Thus with this choice of moving only frequency-independent parts of the selfenergy into the denominator, there remain some quadratic terms in the action. But by construction, these terms vanish on the Fermi surface (see (116)), and hence do not cause divergences.

The choice (114) leads to an adaptive scale decomposition because χ is supported in the neighbourhood

$$\{k : |e(s, k)| \leq \epsilon_s\} \quad (117)$$

of the set $\{k : e(s, k) = e_0(k) + \sigma_s(0, k) = 0\}$ which one may regard as a flowing Fermi surface. In the standard, nonadaptive scale decomposition, the cutoff function is taken as $\chi_0(s, k) = \tilde{\chi}(e_0(p)^2 \epsilon_s^{-2})$, whose support shrinks to the free Fermi surface as $s \rightarrow \infty$. Consequently, the nonadaptive scale decomposition can be used only if a counterterm is included [1, 2] – or if the Fermi surface shift is ignored, as is sometimes the case in approximate studies.

In the present case, the terms in (43) are

$$\begin{aligned} V^{(m)}(\Psi) &= \int \prod_{n=1}^m d\mathbf{k}_n d\mathbf{k}'_n \bar{\psi}(\mathbf{k}_n) \psi(\mathbf{k}'_n) \delta(\sum (\mathbf{k}_n - \mathbf{k}'_n)) \\ &V_m(s \mid \mathbf{k}_1, \dots, \mathbf{k}_m; \mathbf{k}'_1, \dots, \mathbf{k}'_{m-1}) \end{aligned} \quad (118)$$

Equation (18) can now be rewritten as

$$\dot{e}(s, p) = Q(s, p) \chi(s, p) \quad (119)$$

with

$$Q(s, p) = \int d\mathbf{k} F(\mathbf{k}) V_2(s \mid (0, p), \mathbf{k}; \mathbf{k}) \quad (120)$$

The fluctuation propagator

$$\begin{aligned} F(\mathbf{k}) &= A(\mathbf{k})^{-1} \frac{\partial}{\partial s} \chi(s, k) \\ &= A(\mathbf{k})^{-1} \tilde{\chi}' \left(\frac{e(s, k)^2}{\epsilon_s^2} \right) \frac{2}{\epsilon_s^2} (e(s, k)^2 + e(s, k) \dot{e}(s, k)) \end{aligned} \quad (121)$$

contains a term proportional to \dot{e} , because of the adaptive nature of the scale decomposition given by this choice of χ .

To discuss the mathematical properties of the resulting equations, it is useful to introduce the “tadpole” operator. For a function $\Phi(p; \mathbf{k})$ it is the linear operator

$$(\mathcal{L}_\Phi f)(p) = \frac{2}{\epsilon_s^2} \int d\mathbf{q} \Phi(p; \mathbf{q}) \tilde{\chi}' \left(\frac{e(s, q)^2}{\epsilon_s^2} \right) \frac{1}{iq_0 - e(s, q)} f(q) \quad (122)$$

acting on functions of q . With

$$\Phi_V(p, \mathbf{k}) = \mathcal{V}(s \mid (0, p), \mathbf{k}; \mathbf{k}) \quad (123)$$

and

$$\Phi_{V,e}(p, k) = \mathcal{V}(s \mid (0, p), k; k) e(k) \quad (124)$$

(119) becomes

$$(1 - \mathcal{L}_{\Phi_{V,e}})(\dot{e}) = \mathcal{L}_{\Phi_V}(e^2). \quad (125)$$

The operator \mathcal{L} depends nonlinearly on e , but it is linear in \dot{e} .

Thus already the differential equation for the change of e involves an operator inversion, due to the additional term in the fluctuation propagator in (121), hence coming from the adapting scale decomposition. The solution strategy involves the following steps.

1. Show that $1 - \mathcal{L}_{\Phi_{V,e}}$ has a bounded inverse on some open set of V and e . This is essentially a restriction on the size of V_2 — it will fail if V_2 gets so large that its backreaction on e becomes of order 1. A sufficient condition is $\|\mathcal{L}\| < 1$ in a suitable norm. This allows to rewrite (125) as

$$\dot{e} = F(e) = (1 - \mathcal{L}_{\Phi_{V,e}})^{-1} \mathcal{L}_{\Phi_V}(e^2). \quad (126)$$

Because \mathcal{L} depends on e , it is necessary to find a set of e on which $\|\mathcal{L}\| \leq \eta < 1$ uniformly. The existence of this set already requires conditions on e_0 ; the set is then essentially a ball around e_0 .

2. The general theory of ordinary differential equations then implies existence and uniqueness of the solution for all s in some bounded interval, provided that $F(e)$ is Lipschitz. Proving that this condition is satisfied is again non-trivial, in fact it is closely related to questions of possible symmetry breaking.
3. To get global existence (i.e. for all s , one needs to prove that the flow actually never leaves the ball around e_0 .
4. In steps 1–3, V was still regarded as fixed; in the true system of equations, V_2 depends on s and the RHS of the equation for \dot{V}_2 depends on e as well. Thus one needs to show that V_2 stays bounded.

There is a similar set of steps for the discrete Fermi surface flow. Steps 1–3 are already nontrivial, but the hard part is including 4, because it requires dealing with the hierarchy as a whole. For the differential equation, an all-order proof that steps 1–4 work can be given along the lines of [10]. A nonperturbative proof, i.e. without truncation on the hierarchy, remains an open problem, mainly because to this

day there are no good nonperturbative bounds for the continuous RG equation [28], which would be needed in step 4. Up to now, this step requires an integration over short scale intervals and corresponding estimates on discrete RG transformations [29]. Such a proof is given in the Wick ordered approach, for two-dimensional Fermi systems, in [3]. In particular, the boundedness of V_2 necessary in step 4 can be shown in the temperature range of the Fermi liquid criterion of [10].

These mathematical points also correspond to physically natural problems. The question of the existence of the inverse of $1 - \mathcal{L}$ is, in physical terms, basic to the consistency of the method itself: the RG flow is parametrized by energy scales on the (changing) kinetic part of the action, i.e. the part quadratic in the fields. The nonquadratic interaction part is not taken into account when labelling the scales. However, it influences the change \dot{e} of the quadratic part, and the equations become inconsistent when this change becomes larger than the maximal kinetic energy scale itself. Note that it is this condition that is important in the flow; the typical interaction energy, measured by the coupling constant, can be much larger than the scale ϵ_s , and this situation indeed occurs at low scales in all nontrivial models because $\epsilon_s \rightarrow 0$ for $s \rightarrow \infty$. It is a nontrivial fact of fermionic field theory that the method can be controlled by convergent expansions even in that situation [29]. The invertibility of $A(s)$ at all scales s is related to this; the s -dependent Fermi surface must never leave the region where $\chi_{<}(\epsilon_s^{-2}e(s, k)^2) \neq 0$, otherwise an inconsistency manifests itself as divergences in the equation. This condition can be satisfied because power counting implies that the selfenergy at scale s is of order ϵ_s times the coupling constant (in fact, improved power counting [2, 10] gives a better s -behaviour of the bound, which is needed to control derivatives of the selfenergy in the flow).

The counterterms used in [2] can be obtained from the adaptive flow: if $\tilde{e}(k) = \lim_{s \rightarrow \infty} e(s, k)$ exists, the difference $\tilde{e} - e_0$ can be taken as the counterterm K of [2]. This can be used to avoid solving the inversion equation [3]. However, the question whether there is a one-to-one relation between the free and the interacting dispersion function and Fermi surface, which is answered by the inversion theorem, is not solved this way.

The need to invert $1 - \mathcal{L}$ for a tadpole operator \mathcal{L} also arises in the 1PI scheme when one reexpresses the single-scale propagator S in terms of the full propagator \dot{G} in the equation for the self-energy Σ [26, 27]. With the tadpole operator defined in a natural way, the equation for the selfenergy in the 1PI hierarchy becomes

$$\dot{\Sigma} = \mathcal{L}_{\Gamma^{(2)}}(S) = \mathcal{L}_{\Gamma^{(2)}}(\dot{G} - G\dot{\Sigma}G) \quad (127)$$

where $\Gamma^{(2)}$ is the 1-irreducible two-particle vertex and G is the full Green func-

tion. This equation can be rewritten as

$$(1 - \mathcal{L}_{\Gamma^{(4)}_{GG}})(\dot{\Sigma}) = \mathcal{L}_{\Gamma^{(4)}}(\dot{G}). \quad (128)$$

More generally, the occurrence of such operators can be expected in any scheme where the propagator changes, be it with an adaptive scale decomposition, as done here, or without (as in the usual 1PI hierarchy).

The flow discussed here can be adapted to a flow where the density, not the chemical potential μ , is kept fixed. This variant was used in the Appendix of [8] to calculate the flow of the Fermi surface at fixed density in the Hubbard model.

4 Strong coupling behaviour in a toy model

In this section, I study the flow equations for a very simple example, namely the integral over one complex variable, corresponding to the partition function of a zero-dimensional complex bosonic field,

$$P_0(h) = \frac{a_0}{2\pi i} \int_{\mathbb{C}} d\bar{\varphi} \wedge d\varphi e^{-a_0|\varphi|^2 - \lambda_0|\varphi|^4 + h\bar{\varphi} + \bar{h}\varphi} \quad (129)$$

with $a_0 > 0$, $\text{Re } \lambda_0 > 0$ (for simplicity, I take λ_0 real in the calculations below.), and h a complex source field. For $\varphi = u + iv$, $d\bar{\varphi} \wedge d\varphi = 2idu dv$. Thus it is simply a two-dimensional integral.

This analysis is motivated by [20], where it was observed that the 1PI RG equations give an accurate solution for a function similar to $P_0(h)$ even for large λ_0 . This is unexpected at first sight because the diagrammatic approach to these equations may suggest that they are useful only for small λ_0 . Moreover, it is interesting because only the combination of RG and irreducibility seemed to induce the good behaviour at large couplings, namely (1) in contrast to the 1PI scheme, the other schemes studied in [20], namely Polchinski's original hierarchy and the Wick ordered hierarchy of [10], were accurate only at small λ_0 , as one would expect from small-coupling schemes and (2) perturbation theory for the 1PI vertices Γ_m fails as badly as in the other schemes when the coupling constant is not small. Thus it is not merely irreducibility, but a different feature of the 1PI RG hierarchy, that makes for the difference.

A closer look at the equations reveals a first crucial feature: the 1PI equation contains the selfenergy σ in the denominator. When λ_0 gets large, σ becomes of order $\lambda_0^{1/2}$ after a very short flow time already, and (because the signs conspire

well in this toy model) the factor λ_0 in the numerator is balanced by a factor λ_0 in the denominator, which prevents the right hand side of the flow equation from getting very large. This suggests that a dynamic adjustment of the propagator, such as discussed in this paper, improves the behaviour of the other hierarchies as well. This is the case, but the above explanation is not the full story, as we shall see in the following.

4.1 Asymptotics of P at large λ_0

Although we have written P_0 as a function of h , it is understood that it also depends on \bar{h} . Indeed, by the $U(1)$ symmetry of the measure it depends only on $\bar{h}h = |h|^2$. A Gaussian transformation gives the representation

$$P_0(h) = a_0 \int_{\mathbb{R}} \frac{dr}{\sqrt{4\pi\lambda_0}} e^{-r^2/4\lambda_0} \frac{1}{a_0 - ir} e^{\frac{|h|^2}{a_0 - ir}} \quad (130)$$

from which analyticity in $|h|$ is obvious. Write

$$P_0(h) = \sum_{m \geq 0} \pi_m(a_0, \lambda_0) |h|^{2m}. \quad (131)$$

Rescaling the integration variable in (129), one gets $\pi_m(a_0, \lambda_0) = \lambda_0^{-\frac{m}{2}} F_m\left(\frac{a_0}{\sqrt{\lambda_0}}\right)$ where $F_m(\xi)$ is analytic in ξ for $|\xi| < 1$. Explicitly,

$$F_m(\xi) = \frac{1}{2(m!)^2} \xi \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \Gamma\left(\frac{m+r+1}{2}\right) \xi^r. \quad (132)$$

This convergent expansion gives the behaviour at large λ_0 as

$$P_0(0) = \pi_0(a_0, \lambda_0) = \frac{\sqrt{\pi}}{2} \frac{a_0}{\sqrt{\lambda_0}} + O\left(\frac{1}{\lambda_0}\right) \quad (133)$$

and

$$\frac{P_0(h)}{P_0(0)} = e^{w_1|h|^2 + w_2|h|^4 + O(|h|^6)} \quad (134)$$

with

$$w_1 = \frac{1}{\sqrt{\pi}} \lambda_0^{-1/2} + O\left(\frac{1}{\lambda_0}\right), \quad w_2 = -\left(\frac{1}{2\pi} - \frac{1}{8}\right) \lambda_0^{-1} + O(\lambda_0^{-3/2}). \quad (135)$$

Thus for large λ_0 , the $2m^{\text{th}}$ moments of the integral, which play the role of the $2m$ -point functions in this toy model, become small. In the following I discuss to what extent this can be reproduced by the RG with propagator adjustment.

4.2 The dynamical RG hierarchy for the toy model

The usual shift gives

$$P_0(h) = e^{|h|^2/a_0} \left(\mu_{a_0^{-1}} * e^{-w} \right) \left(\frac{h}{a_0} \right) \quad (136)$$

with the notations

$$d\mu(\varphi) = \frac{d\bar{\varphi} \wedge d\varphi}{2\pi i} a_0 e^{-a_0|\varphi|^2}, \quad (\mu * F)(\phi) = \int d\mu(\varphi) F(\phi - \varphi) \quad (137)$$

and $w(\varphi) = \lambda_0|\varphi|^4$. As in the general case, write

$$P(s, h) = e^{k(s)+g(s)|h|^2} \left(\mu_{d(s)} * e^{-v(s)} \right) (\sigma(s)h) \quad (138)$$

and impose $k(0) = 0$, $g(0) = a_0^{-1}$, $d_0 = a_0^{-1}$, $\sigma(0) = a_0^{-1}$, $v(0, \varphi) = w(\varphi)$, as well as $\partial P/\partial s = 0$. The RG equations are derived using straightforward differentiation, in analogy to the general case. Expand

$$v(s, \varphi) = \sum_{m \geq 0} v_m(s) |\varphi|^{2m} \quad (139)$$

then the equations read

$$\dot{v}_0 = \dot{k} - dq - f v_1, \quad \dot{v}_1 = q + f(v_1^2 - 4v_2) \quad (140)$$

and for $m \geq 2$

$$\dot{v}_m = f \left(\sum_{\mu=1}^m \mu(m+1-\mu) v_\mu v_{m+1-\mu} - (m+1)^2 v_{m+1} \right) \quad (141)$$

The initial interaction w has no constant and quadratic part. We now impose that the function v has the same properties. by requiring $\dot{v}_0 = 0$ and $\dot{v}_1 = 0$. This gives equations for q and k :

$$q = 4f v_2 \quad \text{and} \quad \dot{k} = dq. \quad (142)$$

Finally, we fix the flow by setting

$$d(s) = a(s)^{-1} \chi(s) \quad (143)$$

where $a(s) > 0$ is to be determined and $\chi(s) > 0$ a given function of s that decreases from $\chi(0) = 1$ to $\chi(1) = 0$ (here we take $s = 1$ as the final value of the

flow parameter. In this simple example, we do not need to make χ depend on a). We require a and χ to be nonnegative to have the same for d , so that the Gaussian measure is really a measure. Now set

$$\dot{a} = -\chi q \quad (144)$$

Then $f = a^{-1}\dot{\chi}$. Note that f is negative because χ is decreasing; the sign of f is, however, unimportant because it never appears in a measure, only in a Laplacian. With our initial condition, v_2 starts out positive; v_2 decreases and a increases in the flow.

The system of equations reads

$$\dot{k} = a^{-2}\chi\dot{\chi}4v_2 \quad (145)$$

$$\dot{a} = -\chi\dot{\chi}4a^{-1}v_2 \quad (146)$$

$$\dot{v}_2 = -9a^{-1}\dot{\chi}v_3 \quad (147)$$

$$\dot{v}_3 = a^{-1}\dot{\chi}(4v_2^2 - 16v_4) \quad (148)$$

$$\dot{v}_4 = a^{-1}\dot{\chi}(12v_2v_3 - 25v_5) \quad (149)$$

$$\dot{v}_m = a^{-1}\dot{\chi}\left(\sum_{\mu=2}^{m-1}\mu(m+1-\mu)v_\mu v_{m+1-\mu} - (m+1)^2v_{m+1}\right) \quad (150)$$

The summation in (150) now excludes $\mu = 1$ and $\mu = m$ because $v_1 = 0$. The equation for a can be rewritten as

$$\frac{\partial}{\partial s}(a(s)^2) = -4v_2(s) \frac{\partial}{\partial s}(\chi(s)^2). \quad (151)$$

The “amputation” factor $\sigma(s)$ is now determined by the analogue of condition (38): putting $\dot{\sigma}\sigma^{-1} - dq = 0$ and inserting (143) and (144) gives $\sigma^{-1}\dot{\sigma} = -a^{-1}\dot{a}$, thus, by the initial condition, $\sigma(s) = a(s)^{-1}$. Moreover, a glance at (145) and (146) reveals that $\dot{k} = -a^{-1}\dot{a}$. Thus, once $a(s)$ is determined,

$$k(s) = -\log \frac{a(s)}{a_0} \quad \text{and} \quad \sigma(s) = a(s)^{-1} \quad (152)$$

in accordance with the general results of Section 2.6.

4.3 Truncations of the hierarchy

There are still infinitely many equations for the v_m . We now study the accuracy of various truncations as a function of $v_2(0) = \lambda$. Suppose we truncate by setting

$v_3 = 0$. Then $v_2(s) = \lambda_0$ is a constant, and (151) integrates to

$$a(s) = \sqrt{a_0^2 + (\chi(0)^2 - \chi(s)^2)4\lambda_0}. \quad (153)$$

Similarly, the equations for k and g , which in this truncation read

$$\dot{k} = \frac{4\lambda \frac{1}{2}(\chi(s)^2)}{a_0^2 + (\chi(0)^2 - \chi(s)^2)4\lambda} \quad \text{and} \quad \dot{g} = \frac{4\lambda_0 \dot{\chi}}{(a_0^2 + (1 - \chi(s)^2)4\lambda_0)^{3/2}}, \quad (154)$$

can easily be solved. Because $\chi(0) = 1$ and $\chi(1) = 0$, the final values $a_1 = a(1)$, $k_1 = k(1)$ and $g_1 = g(1)$ are

$$a_1 = \sqrt{a_0^2 + 4\lambda_0}, \quad k_1 = \log \left[\left(1 + \frac{4\lambda_0}{a_0^2} \right)^{-\frac{1}{2}} \right], \quad g_1 = \frac{a_0}{a_1^2} = \frac{a_0}{a_0^2 + 4\lambda_0}. \quad (155)$$

The thus obtained approximation to $P_0(h)$ is

$$e^{k_1 + g_1 |h|^2 - \frac{\lambda_0}{a_1^4} |h|^4} = \left(1 + \frac{4\lambda_0}{a_0^2} \right)^{-\frac{1}{2}} e^{\frac{a_0}{a_0^2 + 4\lambda_0} |h|^2 - \frac{\lambda_0}{(a_0^2 + 4\lambda_0)^2} |h|^4} \quad (156)$$

By construction, this approximation is, for small λ_0 , exact to first order in λ_0 . But even at very large λ_0

$$\frac{P_0(0)}{e^{k(1)}} = \sqrt{\pi} + O(\lambda_0^{-1/2}). \quad (157)$$

That is, the behaviour as a function of λ_0 comes out correctly as $\sim \lambda_0^{-1/2}$, only the prefactor is too small. Note that $P_0(0)$ vanishes for large λ_0 , so the absolute error $|P_0(0) - e^{k(1)}| = O(\lambda_0^{-1/2})$ for large λ_0 . The difference of the true second moment to the one calculated in this truncation is much bigger:

$$w_1 - g_1 = \frac{1}{\sqrt{\pi\lambda_0}} - \frac{a_0}{a_0^2 + 4\lambda_0}. \quad (158)$$

but the qualitative behaviour, namely the vanishing at large λ_0 , is still reproduced correctly. The fourth order moment has the right scaling as a function of λ_0 , but the coefficient is too large.

The usual Polchinski hierarchy would not have given even a qualitatively correct behaviour of these coefficients. Clearly, the adjustment of the propagator is crucial for large λ_0 because the end result w_1 for the quadratic term is very different from the starting value a_0^{-1} at large λ_0 .

However, the adjustment of the propagator alone is not sufficient for obtaining good asymptotics at large λ_0 because truncations at higher m improve the result only for small λ_0 , but not at large λ_0 . To see this, let us now truncate by setting $v_4 = 0$, so that (148) becomes

$$\dot{v}_3 = 4a^{-1}\dot{\chi}v_2^2. \quad (159)$$

This corresponds to the usual truncation of the Polchinski equation, where the six-point function is replaced by the tree contribution (see Figure 3). used to obtain the one-loop equation for the four-point function. In the general case, (159) is converted into an integral equation which is then substituted back into (147), leading to a flow equation for v_2 that is nonlocal in the scale parameter s . In the toy model, (147) and (159) can be combined to

$$v_2^2\dot{v}_2 = -\frac{9}{4}v_3\dot{v}_3. \quad (160)$$

Because $v_3 < 0$ at small $s > 0$ by (148),

$$v_3 = -\sqrt{\frac{8}{27}(\lambda_0^3 - v_2^3)} \quad (161)$$

With this, (147) becomes

$$\dot{v}_2 = 9a^{-1}\dot{\chi}\sqrt{\frac{8}{27}(\lambda_0^3 - v_2^3)}, \quad (162)$$

and (162) and (151) now become a closed system. Now there is a serious problem: v_2 decreases in the flow because $\dot{\chi}$ is negative, but this decrease never stops: even if $v_2 = 0$, the λ_0 term in (162) keeps \dot{v}_2 negative, and the flow then goes to negative v_2 and becomes unstable. A numerical analysis of higher truncations shows a partial stabilization, but there is still an instability already at moderately large λ_0 . The reason for this behaviour will be discussed in the next subsections.

4.4 Discussion

The above findings for the toy model can be understood easily in more general terms. The lowest truncation of the hierarchy is well-behaved in the toy model because the correction to a_0 has a good sign, i.e. $a(s)$ remains positive for all s because the λ -dependent terms get added to it (in addition, it is reasonably

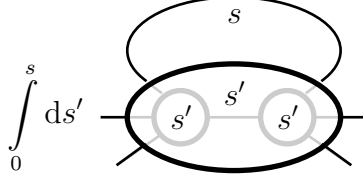


Figure 3: The six-point tree diagram gets integrated over scales $s' < s$ and contracted with a propagator at scale s , to give the right hand side of the flow equation for the four-point function at scale s .

accurate). The higher truncations become unstable at large couplings because the hierarchy is effectively nonlocal in the flow parameter: if one truncates by putting the eight-point function to zero, the right hand side of the flow equation for the six-point function v_3 becomes a tree diagram of four-point functions, so that the equation can be integrated, and the equation for the four-point function v_2 becomes (assuming $v_3(0) = 0$)

$$\frac{\partial}{\partial s} v_2(s, \Psi) = \Delta_{D(s)} \int_0^s ds' v_2(s', \Psi) F(s') v_2(s', \Psi) \quad (163)$$

The graphical representation of this procedure is shown in Figure 3. Because the entire flow history gets inserted on the right hand side, it does not help if v_2 decreases in the flow because there is always the contribution from small s' where $v_2(s')$ is still close to λ_0 , hence large. This is the generalization of the right hand side of (162), in which the λ_0 term kept driving the flow to negative v_2 even at $v_2 = 0$. Thus the nonlocality in the flow parameter makes it impossible to see in this truncation that the effective coupling constant decreases, *but remains positive*, even at large initial coupling. In contrast, the 1PI equation is local in the flow parameter, and the just described “infrared asymptotic freedom” of the toy model at large λ_0 is seen easily. Thus in conclusion, to use the truncated equations at strong coupling it is necessary that the scheme is stable in the sense that asymptotic freedom is not violated in these truncations even at large couplings.

4.5 The Wick ordered flow in the toy model

In the light of the above discussion, one may ask how an adaptive Wick ordered scheme would fare in this example, because the Wick ordered equation also has

the property that it is local in the flow parameter. As shown below, the Wick ordered flow turns out to be well-behaved. In fact, already the initial Wick ordering provides a reasonable approximation to the true result at large λ_0 .

As already discussed in previous sections, Wick ordering has to be done self-consistently if one wants to keep the interaction free of quadratic terms. The initial Wick ordering provides, when done self-consistently, the solution of the mean-field theory (Hartree–Fock equations) as a starting point for the flow. In the toy model, self-consistent Wick ordering means finding a such that

$$a_0|\varphi|^2 + v(|\varphi|) = a|\varphi|^2 + :v(|\varphi|):_{a^{-1}}. \quad (164)$$

Because $v(|\varphi|) = \lambda_0|\varphi|^4$ and $:v(|\varphi|):_{a^{-1}} = e^{-a^{-1}\partial\bar{\partial}}v$, the analogue of (108) is

$$|\varphi|^4 = :|\varphi|^4:_{a^{-1}} + 4a^{-1}|\varphi|^2 - 2a^{-2} \quad (165)$$

Comparison of the quadratic terms in (164) gives the self-consistency condition

$$a_0 = a - 4a^{-1} \lambda_0. \quad (166)$$

The integral becomes

$$P_0(h) = \frac{a_0}{2\pi i} e^{2\lambda_0 a^{-2}} \int_{\mathbb{C}} d\bar{\varphi} \wedge d\varphi e^{-a|\varphi|^2 - : \lambda_0 |\varphi|^4 :_{a^{-1}} + h\bar{\varphi} + \bar{h}\varphi} \quad (167)$$

and the standard shift operations then give

$$P_0(h) = \frac{a_0}{a} e^{2\frac{\lambda_0}{a^2}} e^{\frac{|h|^2}{a}} \int d\mu_{a^{-1}}(\varphi) e^{-\lambda_0 : |\varphi + \frac{h}{a}|^4 :_{a^{-1}}} \quad (168)$$

In contrast to mean-field equations in $d \geq 1$ dimensions, no integrations appear in the self-consistency relation (166). It is simply a quadratic equation with solution

$$\tilde{a} = \frac{a_0}{2} \left(1 + \sqrt{1 + \frac{16\lambda_0}{a_0^2}} \right). \quad (169)$$

For small λ_0 , (169) gives an $O(\lambda_0)$ correction to a_0 , and $e^{2\lambda_0 \tilde{a}^{-2}}$ gives the lowest order correction to $P_0(0)$. For large λ_0 ,

$$\tilde{a} = 2\sqrt{\lambda_0} + \frac{a_0}{2} + \frac{a_0^2}{16\sqrt{\lambda_0}} + O(\lambda_0^{-1}). \quad (170)$$

Originally, the strength of the coupling is given by the “dimensionless” ratio λ_0/a_0^2 . After the initial Wick ordering, this has decreased to

$$\frac{\lambda_0}{\tilde{a}^2} = \frac{1}{4} - O(\lambda_0^{-1}) \quad (171)$$

Thus the “strong coupling problem” in this integral is removed already by the initial Wick ordering. Motivated by this, one can evaluate the remaining convolution integral in (168) to first order in λ_0 , to get the approximation

$$P_0(h) \approx \frac{a_0}{\tilde{a}} e^{2\lambda_0/\tilde{a}^2} e^{|h|^2/\tilde{a} - \frac{\lambda_0}{\tilde{a}^4} |h|^4} \quad (172)$$

At large λ_0 , inserting (170) gives

$$P_0(0) \approx \frac{\sqrt{e}}{2} \frac{a_0}{\sqrt{\lambda_0}}, \quad w_2 \approx \frac{1}{2} \lambda_0^{-1/2} \quad \text{and} \quad w_4 \approx -\frac{1}{16} \lambda_0^{-1} \quad (173)$$

Comparison with (133) and (135) shows that this approximation for $\sqrt{\lambda_0} P_0(0)$ differs from the exact value by 8 %, the one for the prefactor of $\lambda_0^{-1/2}$ in w_1 differs from the exact value by 12 %, and the approximation for the prefactor of λ_0^{-1} in w_2 differs from the true value by less than 50 %. Note that these are the prefactors; the asymptotic behaviour as a function of λ_0 comes out correctly. Thus a simple initial Wick ordering, together with first order perturbation theory, gives a reasonable approximation for $P_0(h)$ even at arbitrarily large λ_0 .

Let us now consider the simplest truncation of the Wick ordered flow with this initial condition. For the toy example, the choice $r = d = a^{-1}\chi$ is possible, so we use this to fix the Wick ordering covariance. It has the advantage that r is determined without any self-consistency argument, but on the other hand, some tadpole terms remain in the equation. Setting $\chi(s) = 1 - s$ with $0 \leq s \leq 1$ and expanding $w(\varphi) = \sum_{m \geq 2} w_m |\varphi|^{2m}$, the RG equations for the Wick ordered functions read

$$\dot{a}(s) = -(1-s) q(s) \quad (174)$$

$$\dot{w}_2(s) = -20(1-s)a(s)^{-2} w_2(s)^2 \quad (175)$$

$$\dot{k}(s) = -8(1-s)^3 a(s)^{-4} w_2(s)^2 \quad (176)$$

$$\dot{g}(s) = a(s)^{-2} q(s) \quad (177)$$

The initial condition is given by the result of initial Wick ordering as (fixing $a_0 = 1$ in the definition of P and recalling (169))

$$a(0) = \tilde{a} = \frac{1}{2} \left(1 + \sqrt{1 + 16\lambda_0} \right)$$

$$w_2(0) = \lambda_0, \quad k(0) = -\ln \tilde{a} + \frac{2\lambda_0}{\tilde{a}^2}, \quad g(0) = \tilde{a}^{-1}.$$

and the function $q(s)$ is given by

$$q(s) = \frac{24w_2(s)^2 a(s)^{-3}(1-s)^2}{1 + 4a(s)^{-2}(1-s)^2w_2(s)} \quad (178)$$

The numerator is the value – in this toy model – of the two-loop sunset diagram which is the only one contributing to $q(s)$ in the truncation $w_3 = 0$. The denominator in the definition of $q(s)$ comes from a tadpole term, which is there because $\dot{d} - f = -a^{-2}\dot{a}\chi$. This is a special case of the term $(1 - \mathcal{L})^{-1}$ discussed in the previous section. Again, the sign is nice in the toy example — for $w_2 > 0$, the denominator is always at least 1, so this inversion does not introduce singularities. Only the equations (174) and (175) are coupled; once their solution has been obtained, the other two simply follow by integration.

Transform to the “dimensionless” functions

$$\omega(s) = \frac{w_2(s)}{a(s)^2} \quad \text{and} \quad \ell(s) = \ln \frac{a(s)}{a(0)}. \quad (179)$$

Then

$$\dot{\omega}(s) = -\omega(s)^2 - 20(1-s) \Phi[(1-s)^2\omega(s)] \quad (180)$$

with $\Phi(x) = \frac{1+\frac{8}{5}x}{1+4x}$ and $\omega(0) = \frac{\lambda_0}{\tilde{a}^2} = \frac{1}{4} - O(\lambda_0^{-1})$. Given the solution of (180), $a(s) = a(0)e^{\ell(s)}$ is obtained from the solution of

$$\dot{\ell}(s) = -24\omega(s)^2 \frac{(1-s)^3}{1 + 4(1-s)^2\omega(s)}. \quad (181)$$

Equation (180) can be integrated explicitly but it is more instructive simply to note that $\omega(s)$ is decreasing and $\omega(s) > 0$ for all s because $\omega(0) > 0$ and the right hand side of (180) is negative and vanishes for $\omega(s) \rightarrow 0$ (the last property failed for the Polchinski-type equation). Thus $\Phi(x)$ appears only with $x \in [0, \omega(0)]$. With $\omega(0) = 1/4$ this implies $0.7 \leq \Phi(\cdot) \leq 1$, hence

$$14(1-s) \leq -\frac{\dot{\omega}(s)}{\omega(s)^2} \leq 20(1-s). \quad (182)$$

Thus the solution for $\omega(s)$ is bounded above and below by that of the prototypical equation for “asymptotic freedom”. Solving this equation and setting $s = 1$ gives $\frac{4}{41} \leq \omega(1) \leq \frac{4}{29}$. Thus there is no instability in the Wick ordered scheme.

To obtain the coefficient of λ_0^{-1} in w_2 , the correct starting values are $\omega(0) = 1/4$ and $a(0) = 2\lambda_0^{1/2}$. With this, the truncated equations give the approximation

$$w_2 \approx \frac{\omega(1)}{a(0)^2 e^{2\ell(1)}} = \frac{\omega(1)}{4 e^{2\ell(1)}} \lambda_0^{-1} = 0.033 \lambda_0^{-1} \quad (183)$$

which differs from the exact value given in (135) by about 20 %. The solution for g gives $w_1 \approx g(1) = 0.578\lambda_0^{-1/2}$. Comparison with (135) shows a deviation of the coefficient of $\lambda_0^{-1/2}$ of about 2.5%.

In summary, the bad behaviour of the Polchinski-type scheme at large couplings is absent from the Wick ordered adaptive flow. The standard truncation $w_3 = 0$ is stable and reproduces the coefficients of the expansion of the exact result in inverse powers of λ_0 well, i.e. with errors of order 1 – 10 %. Of course, the equation can also be used for all (not only large) values of λ_0 , where it has to be evaluated numerically. The stability properties are independent of λ_0 . The numerical values are in good agreement with the exact result. A similarly good strong coupling behaviour of the Wick ordered functions may also be expected in the more general examples discussed in [20].

4.6 Fermionic RG flows at strong coupling: general case

In this section I briefly discuss the question of well-definedness of the RG flow in fermionic systems at large couplings. It has been stated in many places that the RG hierarchy is a diagrammatic method that cannot be taken as a starting point at strong coupling. The true situation is, however, not really so bad: the very strategy of the RG of integrating out degrees of freedom in small steps makes the RG flow well-defined also at large values of the initial coupling function. This is not a special feature of the differential equations but holds also for discrete RG flows, provided the stepsize is chosen small enough.

To see this, consider a fermionic theory, as defined in the Introduction. By integrating the RGDE over short intervals, or using a discrete transformation, one can show that the generating function

$$\mathcal{W}(C, \mathcal{V})(\phi) = \log \int d\mu_C(\varphi) e^{\lambda \mathcal{V}(\varphi+\phi)} = \log(\mu_C * e^{-\mathcal{V}})(\phi) \quad (184)$$

has the representation

$$\mathcal{W}(C, \mathcal{V}) = \mu_C * \mathcal{V} + \sum_{p=2}^{\infty} \lambda^p \mathcal{W}_p(C, \mathcal{V}, \dots \mathcal{V}) \quad (185)$$

where in \mathcal{W}_p , the argument \mathcal{V} appears p times and

$$\mathcal{W}_p(C, \mathcal{V}_1, \dots, \mathcal{V}_p)(\Psi) = \frac{1}{p!} \sum_{T \in \mathcal{T}_p} \int dP(\sigma) \llbracket e^{\Delta_C[M(\sigma)]} \prod_{t \in T} \Delta_C^{(t)} \prod_{q=1}^p \mathcal{V}_q(\Psi_q) \rrbracket \quad (186)$$

where \mathcal{T}_p is the set of all trees on p vertices, and P is a probability measure (i.e. positive and normalized, $\int dP(\sigma) = 1$), and $M(\sigma)$ is a positive definite $p \times p$ matrix with diagonal entries equal to 1. Both P and M can be written down explicitly, but only the above-mentioned properties matter (see [29], where the above formula is derived and explained in more detail). Eq. (185) can be thought of as a resummation of the perturbation expansion in terms of trees. Because $\frac{1}{p!} |\mathcal{T}_p| = \frac{1}{p!} p^{p-2} \leq e^p$, the sum converges if the contribution of every tree T can be bounded by const^p . For fermions, this can be shown using determinant bounds (see, e.g. [22, 29, 30]). The crucial point for the present discussion is that for a short-range interaction \mathcal{V} sufficient conditions for convergence are that the determinant constant δ_C of C (see [30]) and the decay constant $\alpha_C = \sup_X \int |C(X, Y)| dY$ are so small that $|\lambda| \alpha_C \ll 1$. In a RG flow with step-size ε , the covariance C is split into many small parts C_ε and in every RG step, the C in the above formula is replaced by C_ε . Because $\alpha_{C_\varepsilon} \sim \varepsilon$ and $\delta_{C_\varepsilon} \sim \varepsilon^{1/2}$, the above convergence condition can be satisfied for any λ by making the RG stepsize ε small enough. Thus, for small enough ε the first steps in the RG integration are given by convergent perturbation theory in λ , and therefore the RG flow can be started at large coupling. The really interesting analytical and physical question is whether a large, repulsive local initial interaction indeed decreases in the course of the flow (i.e. it gets screened), and whether the rate of this decrease is so fast that the flow can be controlled in the regime where the local coupling is not yet small, as is the case in the toy model.

One may indeed expect that a strong on-site repulsion gets screened in the course of an RG flow, because even a hard-core condition gets softened, thus effectively weaker when one considers larger blocks on the lattice. Namely, if there is a strong on-site repulsion, double occupancy of single sites is exponentially suppressed, but hopping into a block of m^d lattice sites, $m > 1$, is still possible because of local fluctuations in the number density, even if the overall density is fixed. A standard example for a stochastic system with a hopping term and a hard-core repulsion is the asymmetric exclusion process. It was shown in [31] that averaging in a suitable way effectively allows to *remove* the hard-core constraint in this model and still get precise asymptotics of the decay of correlations. Because the truncations of the correlation hierarchy in this example have the property

that they provide successive upper and lower bounds for the exact solution, one can obtain their asymptotic properties in some detail [31]. For fermion systems, such monotonicity properties are in general not known, nevertheless averaging, such as provided by the RG, seems a very promising strategy to see screening effects. In fact, a low-order evaluation of the flow at very large scales in the two-dimensional Hubbard model shows the screening, as well as the generation of the usual antiferromagnetic interaction term [32].

The above argument is about the issue of the behaviour of an *initial* strong short-range interaction. The “flow to strong coupling” at low scales that has been observed in studies of initially weakly coupled systems is an independent phenomenon (as discussed in [13], such flows may be kept under control for a while by phase space arguments, but eventually, the growth of the coupling function wins). The growth of certain parts of the coupling function in these flows is really due to the emergence of order parameter fluctuations that become more and more long-range, hence lead to fermionic interaction vertices that develop singularities in momentum space at a scale where symmetries get broken. In situations where, e.g. the singular part of the fermionic four-point function can be parametrized by an exchange of a boson that becomes massless at a certain scale, approximations that set the momentum of this boson to zero give a coupling constant of order m_{Boson}^{-2} which “runs to strong coupling” and diverges at a certain scale. Whether the boson can really become massless depends on the situation, in particular the dimensionality of the system. Moreover, in systems with long-range interactions, the Anderson–Higgs mechanism may interfere. However, unless this is the case, the Goldstone theorem implies such singularities whenever continuous symmetries are broken. Because such singularities occur only at points, their effect is very different from that of a strong local interaction, which is equally large everywhere in momentum space.

5 Conclusions

The RG equations developed here allow for a dynamical adjustment of propagators in a convenient and flexible way. In particular, they allow for continuously adjusted scale decompositions, which are useful in Fermi surface studies, when one wants to avoid the use of counterterms, or fix the density (instead of the chemical potential) in RG flows. A numerical study of the adaptive flow equation discussed here, with fixed density, was done for the two-dimensional Hubbard model in the appendix of [8]. A mathematical study for the case of regular Fermi

surfaces is in [3]. Taking the Fermi surface deformation into account is important also for the case of singular Fermi surfaces, where the question of pinning of the Fermi surface at Van Hove singularities can be addressed using such flows. This situation is studied in [5] using counterterms. The adaptive method is expected to lead to useful results also here.

In flows with fermion and boson fields, a generalization of the strategy for the adaptive flow becomes obvious: quartic terms in the fermions contribute to the boson propagators, so the dynamical adjustment of the boson propagator allows to take out also certain terms from the fermionic four-point function. A related method has already been used in the 1PI scheme in [33]; a straightforward generalization of the method developed here will allow to do it in any scheme.

Finally, the role of the propagator adjustment for the strong coupling behaviour of flows was investigated for a very simple toy model. It was shown that the Wick ordered scheme with dynamic propagator adjustment correctly reproduces the asymptotic behaviour for arbitrarily large couplings, and already its simplest truncation gives the coefficients of the leading term in the asymptotic series with an error of order 1 – 10 % (depending on the order of the moment). The analysis made clear that the adjustment of propagators is indeed crucial for the success of a scheme at strong coupling, but that in addition, a stability property of the four-point flow is necessary. Namely, the RG equations must be such that asymptotic freedom holds for the solution also if the flow is started at a large value of the coupling. The Polchinski scheme does not have the second property, even with an adapted propagator, but the 1PI and Wick ordered schemes do.

Needless to say, the toy example is very special because there, the signs work out such that a large initial coupling λ_0 decreases to a very small final value, and the quadratic term a_0 increases to a large final value. Because only the combination λ/a^2 matters, this effectively leads to a very rapid transition to a small-coupling situation in the flow, provided the above-mentioned stability holds. This explains the success of the 1PI and Wick ordered scheme. In the latter, already the initial Wick ordering removes the strong coupling problem right away, if it is done self-consistently. It would be too much to hope for such benign signs in more general (and more interesting) models, but, as discussed in some detail in Section 4.6, the flow equations are well-defined at strong coupling (contrary to what has often been stated in the literature) and there are reasons to be optimistic about screening of initially strong local repulsions in RG flows.

Remarks and Acknowledgements. My research on this topic started in 1998, but this paper has been held back for a long time for various reasons. I would like

to thank Carsten Honerkamp and Walter Metzner for discussions, and the KITP, Santa Barbara, especially the program *Realistic Theories of correlated electron systems* and the Erwin Schrödinger Institut, Vienna, for hospitality and financial support in various stages of the work. During the last year, it was also supported by DFG grant Sa 1362/1–1 and NSERC of Canada.

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